COMBINATORIAL DESCRIPTION OF THE COHOMOLOGY OF THE AFFINE FLAG VARIETY

SEUNG JIN LEE

ABSTRACT. We construct the affine version of the Fomin-Kirillov algebra, called the affine FK algebra, to investigate the combinatorics of affine Schubert calculus for type A. We introduce Murnaghan-Nakayama elements and Dunkl elements in the affine FK algebra. We show that they are commutative as Bruhat operators, and the commutative algebra generated by these operators is isomorphic to the cohomology of the affine flag variety. We show that the cohomology of the affine flag variety is product of the cohomology of an affine Grassmannian and a flag variety, which are generated by MN elements and Dunkl elements respectively. The Schubert classes in cohomology of the affine Grassmannian (resp. the flag variety) can be identified with affine Schur functions (resp. Schubert polynomials) in a quotient of the polynomial ring. Affine Schubert polynomials, polynomial representatives of the Schubert class in the cohomology of the affine flag variety, can be defined in the product of two quotient rings using the Bernstein-Gelfand-Gelfand operators interpreted as divided difference operators acting on the affine Fomin-Kirillov algebra. As for other applications, we obtain Murnaghan-Nakayama rules both for the affine Schubert polynomials and affine Stanley symmetric functions. We also define k-strong-ribbon tableaux from Murnaghan-Nakayama elements to provide a new formula of k-Schur functions. This formula gives the character table of the representation of the symmetric group whose Frobenius characteristic image is the k-Schur function.

1. INTRODUCTION

Schubert calculus is a branch of algebraic geometry introduced by Hermann Schubert, in order to solve various counting problems of projective geometry. Schubert studied geometric objects, now called Schubert varieties, which are certain closed varieties in a Grassmannian. The intersection theory of these varieties can be studied via the product structure of the cohomology of the Grassmannian by the solution of Hilbert's Fifteenth problem. Schubert varieties give Schubert classes, elements in the Schubert basis of the cohomology of the Grassmannian, and the Schur functions are polynomial representatives of the Schubert classes. Therefore, Schur functions play an important rule understanding Schubert calculus. For example, the product structure constants in the cohomology ring of the Grassmannian can be identified with some structure constants of Schur functions. Since then, a lot of mathematicians have defined and studied polynomial representatives for Schubert classes in various cohomological theories of varieties such as Grassmannian and flag varieties.

Affine Schubert calculus is a generalization of Schubert calculus for the affine Grassmannian and the affine flag variety instead of the Grassmannian and the flag

variety. The (equivariant) homology of the affine flag variety and the affine Grassmannian can be identified with the nilHecke ring and the Peterson subalgebra by the work of Kostant, Kumar, and Peterson [19, 32] providing an algebraic framework to understand the homology. The combinatorial theory of the affine Schubert calculus was facilitated and studied extensively since Lam [22] showed that the affine Schur functions and the k-Schur functions are polynomial representatives of the Schubert classes in the cohomology and the homology of the affine Grassmannian associated to $SL(n)$. Note that k-Schur functions are introduced by Lapointe, Lascoux, and Morse during the study of the Macdonald positivity conjecture, suggesting that a full development of the combinatorics in affine Schubert calculus will be substantial to understand the Macdonald theory as well. For an introduction to the affine Schubert calculus, see [27].

Compared to combinatorics for the affine Grassmannian, the situation for the affine flag variety is much less known. The best way (to the author's best knowledge) to describe the cohomology of the affine flag variety in the past literature is by taking the dual of the nilHecke algebra [19] which makes combinatorics rather mysterious. There are several serious obstacles to understand the cohomology of the affine flag variety. The fact that Borel's characteristic map is not surjective for the affine flag variety implies that its cohomology is not generated by degree 1 elements, resulting in the absence of the affine Schubert polynomials and their combinatorics in terms of degree 1 elements.

In this paper, we introduce the affine Fomin-Kirillov algebra (affine FK algebra in short), generalizing the Fomin-Kirillov algebra to affine type A, to describe the cohomology of the affine flag variety. Fomin and Kirillov defined a certain quadratic algebra, also called the Fomin-Kirillov algebra, to better understand the combinatorics of the cohomology ring of the flag variety. They showed that the commutative subalgebra generated by Dunkl elements of degree 1 is isomorphic to the cohomology of the flag variety. Since then, a lot of variations for the quadratic algebra has been studied [13, 14, 15, 16, 17, 18]. For example, there are generalizations of the Fomin-Kirillov algebra for K-theory, quantum, equivariant cohomology and for other finite types.

We introduce higher degree generators called Murnaghan-Nakayama elements in the affine FK algebra. The affine FK algebra gives the Bruhat action on the affine nilCoxeter algebra $\mathbb{A}$, which is isomorphic to the homology of the affine flag variety as a $\mathbb{Q}$-module. We show that Murnaghan-Nakayama elements and Dunkl elements commute with each other as Bruhat operators and show that the algebra generated by these elements as Bruhat operators is isomorphic to the cohomology of the affine flag variety as subalgebras in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$. Our proof identifies three different operators on $\mathbb{A}$, namely: Bruhat operators, cap operators defined by the author [29], and the operators defined by Berg, Saliola and Serrano [4] on the affine nilCoxeter algebra $\mathbb{A}$. The identification combines algebraic, geometric and combinatorial components of affine Schubert calculus. Those three operators will be considered as elements in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$.

Positive integers $n \geq 2$ and $k = n-1$ will be fixed throughout the paper. The coefficient ring of the cohomology is $\mathbb{Q}$, and related combinatorics will be adjusted accordingly although three operators are well-defined over $\mathbb{Z}$.

1.1. **Cap operators.** The set $\{A_w : w \in \widetilde{S}_n\}$ forms a basis of $\mathbb{A}$ (see Section 2.4), where $\widetilde{S}_n$ is the affine symmetric group. There is a coproduct structure on $\mathbb{A}$ defined by

$$\Delta(A_w) = \sum p^w_{u,v} A_u \otimes A_v$$

where the sum is over all $u, v \in \widetilde{S}_n$ satisfying $\ell(w) = \ell(u) + \ell(v)$. Kostant and Kumar [19] showed that $p^w_{u,v}$ is the same as the structure coefficient for the cohomology of the affine flag variety. Note that $p^w_{u,v}$'s are nonnegative integers [10].

For $u \in \widetilde{S}_n$, a *cap operator* D_u is defined by

$$D_u(A_w) = \sum p^w_{u,v} A_v.$$

where the sum is over all $v \in \widetilde{S}_n$ satisfying $\ell(w) = \ell(u) + \ell(v)$. Let ϕ_{id} be the map from $\mathbb{A}$ to $\mathbb{Q}$ by taking the coefficient of A_{id}. Then ϕ_{id} induces a $\mathbb{Q}$-module homomorphism $\phi_{\mathrm{id},*}$ from $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$. Note that the cohomology of the affine flag variety is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$ over $\mathbb{Q}$ [19], and the Schubert basis ξ^w can be considered as an element in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$ defined by $\xi^w(A_v) = \delta_{w,v}$ for all $v \in \widetilde{S}_n$. However, the problem with this description is that there is no natural product structure on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$.

One can avoid this problem by considering cap operators. It is obvious that the image of D_w via $\phi_{\mathrm{id},*}$ is ξ^w so that the cohomology of the affine flag variety can be identified with the subalgebra generated by D_w in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$, which naturally has the product structure by composition. Geometrically, $D_u(A_w)$ can be identified with $\xi^u \cap \xi_w$ where ξ^u is the Schubert class for u in the cohomology, ξ_w is the Schubert class for w in the homology of the affine flag variety, and $\cap$ is the cap product.

1.2. **Bruhat operators.** The affine Fomin-Kirillov algebra is generated by $[ij]$ for $i < j$ with distinct residues modulo n with certain quadratic relations among $[ij]$. An element $[ij]$ can be considered as a (right) *Bruhat action* on the affine nilCoxeter algebra defined by

$$A_w \cdot [ij] = \begin{cases} A_{wt_{ij}} & \text{if} \quad \ell(wt_{ij}) = \ell(w) - 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $w \in \widetilde{S}_n$. For an element $\mathbf{x}$ in the affine FK algebra, define $D_{\mathbf{x}}(A_w) = A_w \cdot \mathbf{x}$ when it is well-defined. In this paper, we only consider elements $\mathbf{x}$ such that $D_{\mathbf{x}}(A_w)$ are well-defined for all $w \in \widetilde{S}_n$. See Section 3 for details.

1.3. **BSS operators.** For $a \in \mathbb{Z}$, let $\mathcal{G}^{(a)}$ be the edge-labelled oriented graph defined in [4] with the affine symmetric group as a vertex set: there is an edge from x to y labelled by $y(j) = x(i)$ whenever $\ell(x) = \ell(y) + 1$ and there exists $i \leq a < j$ such that $yt_{ij} = x$. When there is an edge between x and y, the pair (x, y) is called the marked strong order in [25]. Denote $\mathcal{G}^{(0)}$ by $\mathcal{G}$. The *ascent composition* of a sequence $a_1, a_2, \ldots, a_m$ is the composition $[i_1, i_2 - i_1, \ldots, i_j - i_{j-1}, m - i_j]$, where

$i_1 < i_2 < \cdots < i_j$ are the ascents of the sequence.

If $w_0 \xrightarrow{a_1} \cdots \xrightarrow{a_m} w_m$ is a path in $\mathcal{G}^{(a)}$, we denote it by $\operatorname{ascomp}(w_0 \xrightarrow{a_1} \cdots \xrightarrow{a_m} w_m)$ the ascent composition of the sequence of labels $a_1, \dots, a_m$. For a composition $J = [j_1, \dots, j_l]$ of positive integers, let $|J|$ be the sum $j_1 + \cdots + j_l$. Let $D_J^{(a)}$ be the operator in [4] defined by

$$D_J^{(a)}(A_w) = \sum_{\operatorname{ascomp}(w_0 \xrightarrow{a_1} \cdots \xrightarrow{a_m} w_m) = J} A_{w_m}.$$

where the sum runs over all path in $\mathcal{G}$ of length $|J|$ starting at $w = w_0$ whose sequence of labels has the ascent composition J. We call $D_J^{(a)}$ the *BSS operator*. We denote $D_J^{(0)}$ by D_J. When we use the BSS operators $D_J^{(a)}$ in this paper, either $a = 0$ or $J = [1]$.

Berg, Saliola and Serrano [3] showed that $D_{[1]}^{(a)}$ is the same as the cap operator D_{s_a} for $0 \leq a < n$. In [29], the author proved that for a positive integer m, a BSS operator $D_{[m]}$ for $[m]$ is the same as the cap operator $D_{s_{m-1}s_{m-2}\dots s_1 s_0}$, which implies the Pieri rule for the cohomology of the affine flag variety. In this paper, we will abuse notation D_u so that it means one of the above operators depending on whether u is an element in the affine FK algebra, an element in $\widetilde{S}_n$ or a composition.

Now we are ready to state the main theorems in this paper.
For $0 \leq i < m < n$, let $\rho_{i,m}$ be the element $s_{-i}s_{-i+1}\dots s_{-1}s_{m-1-i}s_{m-2-i}\dots s_1 s_0$ in $\widetilde{S}_n$, and $J_{i,m}$ be $[m-i, 1^i]$. Let $\tilde{\theta}_i$ be the Dunkl elements in the affine FK algebra defined in Section 4.1, and let $\mathbf{p}_m$ be the Murnaghan-Nakayama element in the affine FK algebra defined in Section 4.2.

Theorem 1.1. *For $0 \leq i < m < n$, we have*

$$D_{\tilde{\theta}_i} = D_{s_{i+1}} - D_{s_i} = D_{[1]}^{(i+1)} - D_{[1]}^{(i)},$$
$$D_{\mathbf{p}_m} = \sum_{i=0}^{m-1} (-1)^i D_{\rho_{i,m}} = \sum_{i=0}^{m-1} (-1)^i D_{J_{i,m}}.$$

Theorem 1.1 allows us to identify $D_{\tilde{\theta}_i}$ with $\xi^{i+1} - \xi^i$, and $D_{\mathbf{p}_m}$ with $\xi(m) := \sum_{i=0}^{m-1}(-1)^i \xi^{\rho_{i,m}}$. Note that the first equation is immediate from the Chevalley rule and results in [3], so we mainly focus on properties of $D_{\mathbf{p}_m}$ in this paper.

Let $\hat{Fl}, \hat{Gr}, Fl_n$ denote the affine flag variety, the affine Grassmannian and the flag variety. Let $\mathcal{R}$ be the subalgebra in $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ generated by $D_{\tilde{\theta}_i}$ and $D_{\mathbf{p}_m}$ for all i and m. Let $\mathcal{R}_{Fl_n}$ (resp. $\mathcal{R}_{\hat{Gr}}$) be the subalgebra generated by all $D_{\tilde{\theta}_i}$ (resp. all $D_{\mathbf{p}_m}$). Since $\xi(m)$'s generate $\mathcal{R}_{\hat{Gr}}$ and $\xi^{i+1} - \xi^i$ for all i generate $\mathcal{R}_{Fl_n}$, we have the following theorem.

Theorem 1.2. *Subalgebras $\mathcal{R}, \mathcal{R}_{\hat{Gr}}, \mathcal{R}_{Fl_n}$ of $Hom_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ are isomorphic to the cohomology of the affine flag variety, affine Grassmannian, or the flag variety respectively.*

In fact, we show that

$$H^*(\hat{Fl}, \mathbb{Q}) \cong \mathcal{R} \cong \mathcal{R}_{\hat{Gr}} \otimes_{\mathbb{Q}} \mathcal{R}_{Fl_n} \cong H^*(\hat{Gr}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(Fl_n, \mathbb{Q}).$$

See Section 2.4 and Remark 2.2 for more details.

Let R_n be the commutative algebra generated by variables x_i and p_m for $i \in \mathbb{Z}/n\mathbb{Z}$ and $m = 1, \ldots, n-1$ modulo relations $e_j(x_0, \ldots, x_{n-1}) = 0$ for all $j \geq 1$ where e_j is an elementary symmetric function of degree j. Theorem 1.2 implies that R_n is isomorphic to the cohomology $H^*(\hat{Fl})$. Note that the subalgebra $R_n^{\hat{Gr}}$ generated by p_m is isomorphic to $H^*(\hat{Gr}, \mathbb{Q}) \cong \mathcal{R}_{\hat{Gr}}$, and the subalgebra $R_n^{Fl_n}$ generated by x_i is isomorphic to $H^*(Fl_n, \mathbb{Q}) \cong \mathcal{R}_{Fl_n}$ by identifying $D_{\mathbf{p}_m}$ with p_m and $D_{\tilde{\theta}_i}$ with x_i. To sum up, the cohomology of the affine flag variety is isomorphic to

$$R_n = \mathbb{Q}[p_1, \ldots, p_{n-1}, x_0, \ldots, x_{n-1}]/\langle e_j(x_0, \ldots, x_{n-1}), j > 0\rangle \cong R_n^{\hat{Gr}} \otimes_{\mathbb{Q}} R_n^{Fl_n}.$$

1.4. **Affine Schubert polynomials.** Affine Schubert polynomials are elements in R_n representing Schubert classes ξ^w. There are many ways to define the affine Schubert polynomial, but we use divided difference operators to define the affine Schubert polynomials in this paper since the affine FK algebra naturally has divided difference operators and the original Schubert polynomials defined by Lascoux and Schützenberger [24] are defined in terms of divided difference operators. In this paper, we discuss the following about the affine Schubert polynomials.

(1) The definition of the affine Schubert polynomials in terms of divided difference operators.
(2) The relation between the affine Schubert polynomials, the affine Stanley symmetric functions, and Schubert polynomials.
(3) Murnaghan-Nakayama rules and the Pieri rule [29] for the affine Schubert polynomials.

Other definitions, properties and generalizations of the affine Schubert polynomials will be discussed in upcoming joint work with Lam and Shimozono [28]. Since the affine Schubert polynomials contains the Schubert polynomials and affine Stanley symmetric functions, one should try to generalize the known theory for the Schubert polynomials or the affine Stanley symmetric functions to those for the affine Schubert polynomials. It would be very interesting if some of theorems for Schubert polynomials and the affine Stanley symmetric functions can be unified in terms of the affine Schubert polynomials.

In this subsection, we provide the definition of the affine Schubert polynomials.

For $i < j$ with distinct residues mod n, there is a divided difference operator $\Delta_{i,j}$ acting on the affine FK algebra which naturally encodes the BGG operator ∂_i [1] acting on the cohomology of the affine flag variety defined by

$$\partial_i \xi^w = \begin{cases} \xi^{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.3. *For $i \in \mathbb{Z}/n\mathbb{Z}$, the Weyl group action s_i and the divided difference operator $\partial_i := \frac{1-s_i}{x_i - x_{i+1}}$ on R_n can be unquely defined by the following rules.*

(1) For $f, g \in H^*(\hat{Fl})$, we have $s_i(fg) = s_i(f)s_i(g)$. Therefore ∂_i satisfies the Leibniz's rule: for $f, g \in H^*(\hat{Fl})$, we have

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).$$

(2) For nonzero i and for all m, we have

$$s_i(p_m) = p_m$$
$$\partial_i(p_m) = 0.$$

(3) For $i = 0$, we have

$$s_0(p_m) = p_m + x_1^m - x_0^m$$
$$\partial_0(p_m) = \sum_{j=0}^{m-1} x_1^{m-1-j} x_0^j$$

(4) For all $i, j \in \mathbb{Z}/n\mathbb{Z}$, we have

$$s_i(x_j) = x_{s_i(j)}$$
$$\partial_i(x_j) = \delta_{ij} - \delta_{i,j+1}.$$

Theorem 1.4. *The divided difference operators ∂_i on R_n is the same as the BGG operators under the isomorphism $R_n \cong H^*(\hat{Fl})$.*

Now we are ready to define the affine Schubert polynomials.

Definition 1.5. *For $w \in \widetilde{S}_n$, the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ is the unique homogeneous element of degree $\ell(w)$ in R_n satisfying*

$$\partial_i \tilde{\mathfrak{S}}_w = \begin{cases} \tilde{\mathfrak{S}}_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $i \in \mathbb{Z}/n\mathbb{Z}$, with the initial condition $\tilde{\mathfrak{S}}_{id} = 1$.

The affine Schubert polynomials behave surprisingly well with the affine Stanley symmetric functions $\tilde{F}_w$ [22] for $w \in \widetilde{S}_n$ and the Schubert polynomials $\mathfrak{S}_v$ for $v \in S_n$. First note that the divided difference operators ∂_i for nonzero i on $R_n^{Fl_n}$ is the same as the divided difference operators defined by Lascoux and Schützenberger [24], so that $\tilde{\mathfrak{S}}_w$ for $w \in S_n$ is the Schubert polynomial $\mathfrak{S}_w$. Moreover the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ for 0-Grassmannian element w is the same as the affine Schur functions, and for $w \in \widetilde{S}_n$ the projection from R_n to R_n^{Gr} sends $\tilde{\mathfrak{S}}_w$ to the affine Stanley symmetric functions $\tilde{F}_w$ [22].

1.5. **Other applications and remarks.** Fomin and Kirillov [8] conjectured that the Schubert polynomial as an element in FK_n can be written as a nonnegative linear combination of noncommutative variables $[i_1 j_1] \ldots [i_l j_l]$ which is called the nonnegativity conjecture. They also showed that a combinatorial formula of such a polynomial provides a combinatorial formula of the Littlewood-Richardson coefficients for the flag variety. We can generalize the nonnegativity conjecture for the affine Schubert polynomials. Assuming that the subalgebra generated by Dunkl elements and MN elements is isomorphic to the cohomology of the affine flag variety, there is an affine Schubert element $\tilde{\mathfrak{S}}_w^{\widetilde{FK}_n}$ in $\widetilde{FK}_n$ corresponding to the Schubert class ξ^w for $w \in \widetilde{S}_n$. We conjecture that the affine Schubert element can be written as a nonnegative linear combination of noncommutative variables in $\widetilde{FK}_n$. A combinatorial formula for the affine Schubert element would provide a combinatorial formula for the structure constants of the affine Schubert polynomials, which are

the structure constants of the cohomology of the affine flag variety.

For other applications, we obtain Murnaghan-Nakayama rules both for affine Schubert polynomials and affine Stanley symmetric functions. We also provide a marking-free definition of k-Schur functions in terms of power sum symmetric functions p_m using k-strong-ribbon tableaux. Note that the definition of k-Schur functions using strong strips depends on markings, which was one of the major obstacles defining strong strips as elements in the affine FK algebra. Bandlow, Schilling, Zabrocki studied the k-weak-ribbons, called k-ribbons in [2], to describe the MN rule for the k-Schur functions. k-strong-ribbons are combinatorial objects dual to k-weak-ribbons, since k-weak-ribbons give the MN rule for k-Schur functions and k-strong-ribbons give a MN rule for affine Stanley symmetric functions. We also define k-strong-ribbon tableaux from Murnaghan-Nakayama elements to provide a new formula of k-Schur functions. This formula gives the character table of the representation of the symmetric group whose Frobenius characteristic is the k-Schur function.

There are a lot of potential generalizations and applications of the affine FK algebra. Basically, any generalization of the Fomin-Kirillov algebra can be applied to the affine FK algebra. For example, as the Fomin-Kirillov algebra can be generalized to its equivariant, quantum, K-theory version, one may expect that the affine FK algebra can be generalized in a similar way.

The Fomin-Kirillov algebra also has generalizations related with classical Coxeter groups [14]. Kirillov and Maeno defined the bracket algebra and Dunkl elements so that the subalgebra generated by Dunkl elements is canonically isomorphic to the coinvariant algebra of the classical Coxeter groups and $I_2(m)$. Note that the infinite relation (d) defining the affine FK algebra is an analogue of the quadratic relations defining the bracket algebra. Due to this similarity, there is a possibility that the work of Kirillov and Maeno can be generalized to arbitrary infinite Coxeter groups, by combining the theory of the affine FK algebra and the bracket algebra.

It would be very interesting to relate the affine FK algebra with the affine Grothendieck polynomials defined by Kashiwara and Shimozono [12], which are certain rational functions that represent the Schubert classes in the K-theory of the affine flag variety up to localization. There is also a chance that the affine FK algebra have a t-dependent version that connects with t-dependent affine Schubert calculus. Defining t-dependent affine Schubert polynomials generalizing t-dependent affine Schur functions [7] and relating them with branching of t-dependent k-Schur functions [26] is one research direction.

The paper is structured as follows: In Section 2, we review affine symmetric groups, symmetric functions, affine flag varieties, affine Grassmannians, affine nil-Coxeter algebra, and the Fomin-Kirillov algebra. In Section 3, we define the affine FK algebra and study the Bruhat operator acting on the affine nilCoxeter algebra. In Section 4, we define Dunkl elements and Murnaghan-Nakayama elements, and investigate relations among those elements. In Section 5, we derive identities that uniquely determine Bruhat operators for MN elements. In Section 6, we recall

properties of cap operators and BSS operators and prove Theorem 1.1, 1.2. In Section 7, we define the divided difference operators acting on the affine FK algebra and those operators are essentially the same as the BGG operators. In Section 8, we define and discuss the affine Schubert polynomials. In Section 9, we state the nonnegativity conjecture in the affine FK algebra. In section 10, we state the Murnaghan-Nakayama rules both for the affine Schubert polynomial and the affine Stanley symmetric functions. In Section 11, we discuss a new formula for k-Schur functions in terms of power sum symmetric functions as well as its relation with representation theory.

2. Preliminaries

2.1. **Affine symmetric group.** Let I be the set $\{0,1,\ldots,n-1\} = \mathbb{Z}/n\mathbb{Z}$. Let $\widetilde{S}_n$ denote the affine symmetric group with simple generators $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\ s_i s_j &= s_j s_i && \text{if } i-j \neq 1, -1. \end{aligned}$$

where indices are taken modulo n. An element of the affine symmetric group may be written as a word in the generators s_i. A *reduced word* of the element is a word of minimal length. The *length* of w, denoted $\ell(w)$, is the number of generators in any reduced word of w.

The *Bruhat order*, also called *strong order*, on affine symmetric group elements is a partial order where $u < w$ if there is a reduced word for u that is a subword of a reduced word for w. If $u < w$ and $\ell(u) = \ell(w) - 1$, we write $u \lessdot w$. It is well-known that $u \lessdot w$ if and only if there exists a transposition t_{ab} in $\widetilde{S}_n$ such that $w = ut_{ab}$ and $\ell(u) = \ell(w) - 1$. See [5] for instance.

The subgroup of $\widetilde{S}_n$ generated by $\{s_1, \cdots, s_{n-1}\}$ is naturally isomorphic to the symmetric group S_n. The 0-Grassmannian elements are minimal length coset representatives of $\widetilde{S}_n/S_n$. In other words, w is 0-Grassmannian if and only if all reduced words of w end with s_0. More generally, for $i \in \mathbb{Z}/n\mathbb{Z}$ and $w \in \widetilde{S}_n$, w is called i-Grassmannian if all reduced words of w end with s_i. We denote the set of i-Grassmannian elements by $\widetilde{S}_n^i$.

For $a \in \mathbb{Z}$ and $u, w \in \widetilde{S}_n$, a *marked strong cover* ($u \xrightarrow{(j_1,j_2)} w$) with respect to a is defined by $w \lessdot wt_{j_1,j_2} = u$ with $j_1 \leq a < j_2$ and $\ell(w) = \ell(u) - 1$. For this cover, we distinguish two strong covers corresponding to t_{j_1,j_2} and t_{j_1+n,j_2+n} so that we may have multiple covers between fixed w, u. We call (j_1, j_2) the *index* of the marked strong cover. Note that $w(j_2)$ is called the marking of the strong cover in [3, 4, 25] and the marking is used to define strong strips and k-Schur functions.

2.2. **Symmetric functions.** Let Λ denote the ring of symmetric functions over $\mathbb{Q}$. For a partition λ, we let $m_\lambda, h_\lambda, p_\lambda, s_\lambda$ denote the monomial, homogeneous, power sum and Schur symmetric functions, respectively, indexed by λ. Each of these families forms a basis of Λ. Let $\langle \cdot, \cdot \rangle$ be the Hall inner product on Λ satisfying

$\langle m_\lambda, h_\mu\rangle = \langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}$ for partitions λ, μ.

Let $\Lambda_{(k)}$ denote the subalgebra generated by $h_1, h_2, \dots, h_k$. The elements h_λ with $\lambda_1 \le k$ form a basis of $\Lambda_{(k)}$. We call a partition λ with $\lambda_1 \le k$ a *k-bounded partition*. Note that there is a bijection between the set of k-bounded partitions and the set of 0-Grassmannian elements in $\widetilde{S}_n$ [25]. We denote by $\lambda(w)$ the k-bounded partition corresponding to the 0-Grassmannian element w. Let $\Lambda^{(k)} = \Lambda/I_k$ denote the quotient of Λ by the ideal I_k generated by m_λ with $\lambda_1 > k$. The image of the elements m_λ with $\lambda_1 \le k$ form a basis of $\Lambda^{(k)}$. Note that I_k is isomorphic to the ideal generated by p_λ for $\lambda_1 > k$, so that p_λ for k-bounded partitions λ form a basis of $\Lambda^{(k)}$.

There is another remarkable basis for $\Lambda_{(k)}$ and $\Lambda^{(k)}$. For a k-bounded partition λ, a k-Schur function $s_\lambda^{(k)}$ and an affine Schur function $\tilde{F}_\lambda$ are defined in [21, 25]. Lam [22] showed that the k-Schur functions (resp. the affine Schur functions) are representatives of the Schubert basis of the homology (resp. the cohomology) of the affine Grassmannian $\hat{Gr}$ for $SL(n)$ via the isomorphism of Hopf-algebras

$$\begin{aligned}\Lambda_{(k)} &\cong H_*(\hat{Gr})\\ \Lambda^{(k)} &\cong H^*(\hat{Gr}).\end{aligned}$$

The restriction of the Hall inner product on $\Lambda^{(k)} \times \Lambda_{(k)}$ gives the identity $\langle \tilde{F}_\lambda, s_\mu^{(k)}\rangle = \delta_{\lambda,\mu}$. Since we do not use the definitions of k-Schur functions, affine Schur functions, and affine Stanley symmetric functions in this paper, definitions are omitted. See [21, 22, 25] for more details.

2.3. **Affine flag varieties and affine Grassmannians.** We define the Kac-Moody flag variety $\hat{Fl}$, the affine Grassmannian and the Schubert basis on the (co)homology of the affine flag variety and the affine Grassmannian. There are two definitions of the Kac-Moody flag variety $\hat{Fl}$ in [27], but we only recall $\hat{Fl}$ as the Kac-Moody flag ind-variety in [19, 20].

Let G_{af} denote the Kac-Moody group of affine type associated with $G := SL(n)$ and let B_{af} denote its Borel subgroup. The Kac-Moody flag ind-variety $\hat{Fl} = G_{\mathrm{af}}/B_{\mathrm{af}}$ is paved by cells $B_{\mathrm{af}}\dot{w}B_{\mathrm{af}}/B_{\mathrm{af}} \cong \mathbb{C}^{\ell(w)}$ whose closure X_w is called the Schubert variety. A Schubert variety defines a Schubert class $\xi^w \in H^*(\hat{Fl})$ and $\xi_w \in H^*(\hat{Fl})$. The affine Grassmannian $\hat{Gr}$ is $G_{\mathrm{af}}/P_{\mathrm{af}}$ where P_{af} is the maximal paraboic subgroup obtained by "omitting the zero node".

There are isomorphisms

$$H_*(\hat{Fl}) \cong H_*(\hat{Gr}) \otimes_{\mathbb{Q}} H_*(Fl_n), \tag{1}$$

$$H^*(\hat{Fl}) \cong H^*(\hat{Gr}) \otimes_{\mathbb{Q}} H^*(Fl_n). \tag{2}$$

which is clear from Theorem 2.1, where Fl_n is the flag variety. Let us describe projections and inclusions between the (co)homology of $\hat{Fl}, \hat{Gr}$ and Fl_n following Peterson's work [32].

Let G be $SL(n)$, B Borel subgroup, T the maximal torus, K the maximal compact form of G, $T_{\mathbb{R}} = K \cap T$, ΩK the topological group of based loops into K, and LK the space of loops into K. There are weak homotopy equivalences $\Omega K \cong \hat{Gr}$ and $LK/T_{\mathbb{R}} \cong \hat{Fl}$ under which the actions of $T_{\mathbb{R}}$ and T correspond. Then there is an inclusion $p_1 : \Omega K \to LK/T_{\mathbb{R}}$, called the wrong way map. Lam showed that the affine Stanley symmetric function $\tilde{F}_w$ is the pullback $p_1^*(\xi^w)$ in $H^*(\hat{Gr}) \cong \Lambda^{(k)}$ [22]. Note that we also have a surjection $q_1 : \hat{Fl} \to \hat{Gr}$ such that $q_1 \circ p_1$ is the identity up to homotopy.

We also have an inclusion $p_2 : K/T_{\mathbb{R}} \to LK/T_{\mathbb{R}}$ by sending an element in K to the constant loop, as well as the evaluation map $ev : LK/T_{\mathbb{R}} \to K/T_{\mathbb{R}}$ at the identity. The evaluation map induces the surjection $p_2^* : H^*(Fl_n) \to H^*(\hat{Fl})$ mapping the Schubert class $\xi^{s_i}_{Fl_n}$ for s_i with $1 \leq i \leq n-1$ to $\xi^{s_i} - \xi^{s_0}$. The identity $p_2^*(\xi^{s_i}_{Fl_n}) = \xi^{s_i} - \xi^{s_0}$ will be proved in Section 2.4.

2.4. **Affine nilCoxeter algebra.** In this subsection, we review the theory of the affine nilCoxeter algebra and its connections with the previous section.

The *affine nilCoxeter algebra* $\mathbb{A}$ is the algebra generated by $A_0, A_1, \ldots, A_{n-1}$ over $\mathbb{Z}$, satisfying

$$\begin{aligned} A_i^2 &= 0 \\ A_iA_{i+1}A_i &= A_{i+1}A_iA_{i+1} \\ A_iA_j &= A_jA_i \qquad\qquad \text{if } i-j \neq 1, -1. \end{aligned}$$

where the indices are taken modulo n. The subalgebra $\mathbb{A}_f$ of $\mathbb{A}$ generated by A_i for $i \neq 0$ is isomorphic to the nilCoxeter algebra studied by Fomin and Stanley [9]. The simple generators A_i are considered as the *divided difference operators* (see Section 7).

The A_i satisfy the same braid relations as the s_i in $\widetilde{S}_n$, i.e., $A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$. Therefore it makes sense to define

$$\begin{array}{rcll} A_w &=& A_{i_1}\cdots A_{i_l} & \text{where} \\ w &=& s_{i_1}\cdots s_{i_l} & \text{is a reduced decomposition.} \end{array}$$

One can check that

$$A_vA_w = \begin{cases} A_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

A word $s_{i_1}s_{i_2}\cdots s_{i_l}$ with indices in $\mathbb{Z}/n\mathbb{Z}$ is called *cyclically decreasing* if each letter occurs at most once and whenever s_i and s_{i+1} both occur in the word, s_{i+1} precedes s_i. For $J \subsetneq \mathbb{Z}/n\mathbb{Z}$, a *cyclically decreasing element* w_J is the unique cyclically decreasing permutation in $\widetilde{S}_n$ which uses exactly the simple generators in $\{s_j \mid j \in J\}$. For $i < n$, let

$$\mathbf{h}_i = \sum_{\substack{J \subset \mathbb{Z}/n\mathbb{Z} \\ |J| = i}} A_{w_J} \in \mathbb{A}$$

where $\mathbf{h}_0 = 1$ and $\mathbf{h}_i = 0$ for $i < 0$ by convention. Lam [21] showed that the elements $\{\mathbf{h}_i\}_{i<n}$ commute and freely generate a subalgebra $\mathbb{B}$ of $\mathbb{A}$ called the *affine Fomin-Stanley algebra*. It is well-known that $\mathbb{B}$ is isomorphic to $\Lambda_{(k)}$ via the map

sending $\mathbf{h}_i$ to h_i. Therefore, the set $\{\mathbf{h}_\lambda = \mathbf{h}_{\lambda_1} \dots \mathbf{h}_{\lambda_l} \mid \lambda_1 \leq k\}$ forms a basis of $\mathbb{B}$.

There is another basis of $\mathbb{B}$, called the *noncommutative k-Schur functions* $\mathbf{s}_\lambda^{(k)}$. For a bounded partition λ, the noncommutative k-Schur function $\mathbf{s}_\lambda^{(k)}$ is the image of $s_\lambda^{(k)}$ via the isomorphism $\Lambda_{(k)} \cong \mathbb{B}$. It is shown [22] that the noncommutative k-Schur function $\mathbf{s}_\lambda^{(k)}$ is the unique element in $\mathbb{B}$ that has the unique 0-Grassmannian term A_{w_λ} where w_λ is the 0-Grassmannian element corresponding to λ. We also denote $s_\lambda^{(k)}$ by $s_{w(\lambda)}^{(k)}$. The noncommutative k-Schur functions are non-equivariant version of the j functions studied by Peterson [32] for affine type A. For details and the original definition of noncommutative k-Schur functions, see [22, 27].

We recall the following theorem proved in [4].

Theorem 2.1. *For an element $w \in \widetilde{S}_n$, let $w = w_0 w_1$ be the unique decomposition with a 0-Grassmannian element w_0 and $w_1 \in S_n$. Then the set*

$$\{\mathbf{s}_{w_0}^{(k)} A_{w_1} \mid w \in \widetilde{S}_n\}$$

forms a basis of $\mathbb{A}$.

One can show Theorem 2.1 by an induction on $\ell(w_0)$ with the fact that $\mathbf{s}_{w_0}^{(k)}$ has the unique 0-Grassmannian term A_{w_0}.

Since $\mathbb{B}$ (resp. $\mathbb{A}_f$) is isomorphic to the homology of the affine Grassmannian (resp. flag variety), Theorem 2.1 provides combinatorial interpretations of the isomorphism (1) and (2). The pullbacks and pushforwards of maps p_1, q_1, p_2, ev can be purely written in terms of the affine nilCoxeter algebra. For example, the pushforward of the evaluation map is

$$ev_* : \mathbb{A} \to \mathbb{A}_f$$

sending $\mathbf{s}_{w_0}^{(k)} A_{w_1}$ to 0 if w_0 is not the identity, and to A_{w_1} otherwise. The evaluation map also induces the pullback $ev^* : H^*(Fl_n) \to H^*(\hat{Fl})$. In terms of the affine nilCoxeter algebra, the pullback is $ev^* : \mathrm{Hom}(\mathbb{A}_f, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$ by sending $g \in \mathrm{Hom}(\mathbb{A}_f, \mathbb{Q})$ to $ev^*(g)$ defined by $ev^*(g)(\mathbf{s}_{w_0}^{(k)} A_{w_1}) = g(A_{w_1})$ if w_0 is the identity, and $= 0$ otherwise. One can easily show $\mathbf{s}_{s_0}^{(k)} = \mathbf{h}_1 = A_0 + A_1 + \ldots + A_{n-1}$, and the image $ev^*(\xi^{s_i}_{Fl_n})$ for $i \neq 0$ is $\xi^{s_i} - \xi^{s_0}$ since $(\xi^{s_i} - \xi^{s_0})(\mathbf{s}_{s_0}^{(k)}) = 0$ and $(\xi^{s_i} - \xi^{s_0})(A_j) = \delta_{ij}$ for nonzero j. Let $\mathcal{R}_{\hat{Gr}}$ and $\mathcal{R}_{Fl_n}$ be the image $p_1^*(H^*(\hat{Gr}))$ and $ev^*(H^*(Fl_n))$ in $H^*(\hat{Fl})$ respectively. Then we have

$$H^*(\hat{Fl}) \cong \mathcal{R}_{\hat{Gr}} \otimes_{\mathbb{Q}} \mathcal{R}_{Fl_n} \tag{3}$$

Remark 2.2. *Recall that Theorem 1.1 identifies $D_{\tilde{\theta}_i}$ with $\xi^{i+1} - \xi^i$, and $D_{\mathbf{p}_m}$ with $\xi(m) := \sum_{i=0}^{m-1} (-1)^i \xi^{\rho_{i,m}}$. From above computation, $\xi^{i+1} - \xi^i$ for $i \in \mathbb{Z}/n\mathbb{Z}$ generate $\mathcal{R}_{Fl_n}$. Since we identify $\xi(m)$ with p_m in $\Lambda^{(k)} \cong \mathcal{R}_{\hat{Gr}}$ (Section 6) and p_m generate $\Lambda^{(k)}$, $\xi(m)$ generates $\mathcal{R}_{\hat{Gr}}$. Therefore, Theorem 1.2 follows from Theorem 1.1.*

2.5. **Fomin-Kirillov algebra.** We review some facts about the Fomin-Kirillov algebra proved in [8].

Definition 2.3. *For a fixed positive integer n, let FK_n be the the free algebra generated by $\{[ij] : i, j \in \mathbb{Z}, 1 \le i < j \le n\}$ with the following relations:*

$$[ij]^2 = 0.$$
$$[ij][kl] = [kl][ij] \text{ for distinct } i, j, k, l.$$
$$[ij][jk] = [jk][ik] + [ik][ij] \text{ and } [jk][ij] = [ik][jk] + [ik][ij] \text{ for distinct } i, j, k.$$

Let θ_i be the Dunkl elements defined by $\sum_{j \ne i}[ij]$. Here for $i < j$, $[ji]$ is the same as $-[ij]$. Then the θ_i's commute pairwise for all i. Moreover, all symmetric functions in Dunkl elements vanish in FK_n and these are all relations between θ_i. Therefore, the commutative subalgebra generated by Dunkl elements is isomorphic to the cohomology of the flag variety (see [8]).

3. Affine Fomin-Kirillov algebra

For $i \in \mathbb{Z}$, let $\bar{i}$ be the residue of i modulo n.

Definition 3.1. *Let A be the free algebra generated by $\mathcal{S} = \{[ij] : i, j \in \mathbb{Z}, i < j, \bar{i} \ne \bar{j}\}$. Let $A(N)$ be the subalgebra of A generated by elements $[ij]$ with $|i|, |j| \ge N$. Then we have a filtration*

$$A = A(0) \supset A(1) \supset A(2) \supset \cdots .$$

Let $\mathcal{A}$ be the inverse limit $\varprojlim (A/A(i))$.

An element $\mathbf{x}$ in $\mathcal{A}$ can be written as a (possibly infinite) sum $\sum_{J,m} a_{J,m}\mathbf{x}_{J,m}$ where $a_{J,m} \in \mathbb{Z}$, J is in $\prod_{i=1}^m \mathcal{S}$ for $m \ge 0$, and $\mathbf{x}_{J,m} = [j_{1,1}, j_{1,2}][j_{2,1}, j_{2,2}] \cdots [j_{m,1}, j_{m,2}]$ when $J = ([j_{1,1}, j_{1,2}], [j_{2,1}, j_{2,2}], \cdots , [j_{m,1}, j_{m,2}])$. For $m = 0$, J is an empty set and we set $\mathbf{x}_{J,m} = 1$. For $i > j$, we use the convention $[ij] = -[ji]$. We call $\mathbf{x}_{J,m}$ a *noncommutative monomial* in $\mathcal{A}$.

Define the *Bruhat action* of $[ij]$ on $\mathbb{A}$ by

$$A_w \cdot [ij] = \begin{cases} A_{wt_{ij}} & \text{if} \quad \ell(wt_{ij}) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For an element in A, one can define an action on $\mathbb{A}$ extended linearly. For an element $\mathbf{x}$ in $\mathcal{A}$, even if $\mathbf{x}$ is an infinite summation of product of $[ij]$'s, it is possible that all but finitely many terms $A_w \cdot \mathbf{x}_{J,m}$ vanish when acting $\mathbf{x}$ on an element A_w. Let $\mathcal{E}$ be the subalgebra of $\mathcal{A}$ consisting of elements which give a valid action on $\mathbb{A}$. Most elements in $\mathcal{A}$ in this paper contain an infinite sum but have a valid action on $\mathbb{A}$. Define the map $D : \mathcal{E} \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ by sending $\mathbf{x}$ to $D_{\mathbf{x}}$, where $D_{\mathbf{x}}(A_v) := A_v \cdot \mathbf{x}$. We call $D_{\mathbf{x}}$ a *Bruhat operator* for $\mathbf{x}$. We often say "$\mathbf{x}$ as a Bruhat operator" instead of $D_{\mathbf{x}}$ since we are mainly interested in describing the cohomology of the affine flag variety as a subalgebra in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$.

As Bruhat operators on the affine nilCoxeter algebra, we have following relations between the operators $[ij]$.

(a) $[ij]^2 = 0$.

(b) $[ij][kl] = [kl][ij]$ if $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ are all distinct.

(c) For i, j, k with distinct residues, $[ij][jk] = [jk][ik] + [ik][ij]$ and $[jk][ij] = [ik][jk] + [ik][ij]$.

(d) For distinct i, j with $\bar{i} \neq \bar{j}$, $\sum_{\overline{i'}=\bar{i},\overline{j'}=\bar{j}}[ij'][ji'] = 0$.

(e) $[i, j] = [i + n, j + n]$.

Note that the relations (a)-(c) are analogous to those in the definition of the Fomin-Kirillov algebra, and proofs for these relations are similar. The relation (d) is an affine type A analogue of the quadratic relation in the bracket algebra which is a generalization of the Fomin-Kirillov algebra to (classical) Coxeter groups [14]. The relation (e) is obvious since we have $t_{i,j} = t_{i+n,j+n}$ as elements in the affine symmetric group. The quotient algebra of $\mathcal{E}$ modulo relations (a)-(e) is called the *affine Fomin-Kirillov algebra* $\widetilde{FK}_n$.

Note that the Bruhat action of $\widetilde{FK}_n$ on $\mathbb{A}$ is not faithful. One can show that for x, y, z with distinct residues modulo n, we have $[xy][yz][xy] = [yz][xy][yz]$ and this relation is not implied by relations (a)-(e). For later use, define $\mathcal{A}'$ by the quotient algebra of $\mathcal{A}$ modulo relations (a)-(c).

The following lemma is useful to prove that certain infinite expressions in $\mathcal{A}$ give a valid action, thus also in $\mathcal{E}$.

Theorem 3.2. *For a positive integer M, let $\mathcal{S}_M$ be the subset of $\mathcal{S}$ consisting of elements $[ij]$ with $0 < j - i < M$. For an element $\mathbf{x} = \sum_{J,m} a_{J,m}\mathbf{x}_{J,m}$ in $\mathcal{A}$ if for any constant M, all but finitely many $a_{J,m}$ for $J \in \prod_{i=1}^{m} \mathcal{S}_M$ vanish, then the element $\mathbf{x}$ gives a valid action on $\mathbb{A}$.*

Proof. It is enough to show that the element $\mathbf{x}$ as an action on A_w gives a valid element in $\mathbb{A}$. Note that all A_v's appearing in $A_w \cdot \mathbf{x}$ satisfy $w > v$ where $>$ is the Bruhat order in $\widetilde{S}_n$. There are only finitely many chains of Bruhat covers starting from w to v when we identify two covers $w_1 \lessdot w_2$ corresponding to indices (i, j) and $(i + n, j + n)$. Therefore, the set consisting of $j - i$ for all indices (i, j) appearing in the Bruhat interval $[v, w]$ has an upper bound. One can set this upper bound to M and apply the hypothesis to make $A_w \cdot \mathbf{x}$ a finite expression. The theorem follows. $\square$

Corollary 3.3. *For an element $\mathbf{x} = \sum_{J,m} a_{J,m}\mathbf{x}_{J,m}$ in $\mathcal{A}$, if there exist constants a, b such that all $[ij]$'s appearing in the expression satisfy $a < j$ and $i < b$ then $\mathbf{x}$ is in $\mathcal{E}$.*

Proof. Since we have only finitely many $[ij]$'s satisfying $a < j, i < b, j - i < M$ for fixed constants a, b, M, the corollary follows. $\square$

Note that there is a (left) $\widetilde{S}_n$- action on $\widetilde{FK}_n$ (and $\mathcal{E}$) defined by

$$w[ij] = [w(i), w(j)]$$

for $w \in \widetilde{S}_n$ and $[ij] \in \mathcal{S}$. Indeed, one can check that the two-sided ideal generated by relations (a)-(e) is invariant under the $\widetilde{S}_n$-action.

4. Dunkl elements and Murnaghan-Nakayama elements

In this section, we define Dunkl elements and MN elements and investigate identities among these elements.

4.1. Dunkl elements. For $i \in \mathbb{Z}$, a Dunkl element $\tilde{\theta}_i$ can be defined in an analogous way to the definition of the Dunkl element in FK_n defined in Section 2.5.

Definition 4.1. *For $i \in \mathbb{Z}$, define a Dunkl element $\tilde{\theta}_i$ by $\sum_{j\in\mathbb{Z},\bar{j}\neq\bar{i}}[ij]$.*

Note that all terms appearing in $\tilde{\theta}_i$ are either of the form $[ij]$ for $i < j$, or $[ji]$ for $j < i$. Therefore, $\tilde{\theta}_i$ satisfies the hypothesis in Corollary 3.3 with $a = i-1$ and $b = i+1$, thus it gives a well-defined action on $\mathbb{A}$. Note that the Dunkl elements in $\widetilde{FK}_n$ generalize those defined in FK_n.

In this subsection, we use relations (a)-(e) to prove properties of Dunkl elements.

Theorem 4.2. *$\tilde{\theta}_i$'s commute with each other for all i.*

Proof. For fixed i, j with distinct residues, there are 3 cases for terms appearing in $\tilde{\theta}_i\tilde{\theta}_j$. For each term $[ix][jy]$ (or $[jy][ix]$), let the support of the term be the set $\{i, x, j, y\}$ modulo n. If the cardinality of the support is 4, then we have $[ix][jy] = [jy][ix]$ by the relation (a). Hence the restriction of $\tilde{\theta}_i\tilde{\theta}_j - \tilde{\theta}_j\tilde{\theta}_i$ on any support with cardinality 4 vanishes.
If the cardinality of the support is 3, $\tilde{\theta}_i\tilde{\theta}_j - \tilde{\theta}_j\tilde{\theta}_i$ vanishes on the support $\{i, j, x\}$ from the identity

$$[ix][jx] + [ij][jx] + [ix][ji] = [jx][ix] + [jx][ij] + [ji][ix].$$

The above identity follows from the relation (b).
If the cardinality of the support is 2, the support must be $\{i, j\}$ modulo n. Then $\tilde{\theta}_i\tilde{\theta}_j - \tilde{\theta}_j\tilde{\theta}_i$ vanishes on the support by the relation (d). □

The following lemma is useful to derive additional identities among $\tilde{\theta}_i$.

Theorem 4.3. *Let $a, b_1, \dots, b_m$ be distinct integers modulo n. Then we have*

$$\sum\Big([ab_1][ab_2]\dots[ab_m][ab'_1] + [ab_2][ab_3]\dots[ab_m][ab_1][ab'_2] + \dots + [ab_m][ab_1]\dots[ab_{m-1}][ab'_m]\Big) = 0$$

where the sum is over all integers b'_i congruent to b_i modulo n.

Proof. We omit the proof since it is a simple generalization of [8, Lemma 7.2], proved by induction on m. For $m = 1$, the theorem follows from the relations (d) and (e). □

We have the following corollaries of Theorem 4.3, which will be frequently used later.

Corollary 4.4.

$$\tilde{\theta}_i^m = \sum[ia_1]\dots[ia_m]$$

where the sum is over all integers $a_1, \dots a_m$ such that $a_1, \dots, a_m, i$ have distinct residues modulo n.

Corollary 4.5. *If $m \geq n$, then $\tilde{\theta}_i^m = 0$ for all i.*

From now on, we will not need to use the relations (d) and (e) in the definition of $\widetilde{FK}_n$ after using Corollary 4.4, 4.5.

4.2. **Murnaghan-Nakayama elements.** We define Murnaghan-Nakayama elements $\mathbf{p}_m(i)$ (MN elements in short) in $\widetilde{FK}_n$ as a generalization of $\theta_1^m + \cdots + \theta_i^m$ in FK_n. Unlike the finite case, MN elements are not generated by Dunkl elements $\tilde{\theta}_i$. We define MN elements by investigating the combinatorics of the Fomin-Kirillov algebra studied by Mészáros, Panova, Postnikov [30] and generalizing them to affine case.

Let $\mathcal{D}$ be the 2-dimensional infinite grid. A *box* is specified by its position (i, j) when the vertices of the box are $(i, j), (i, j+1), (i+1, j), (i+1, j+1)$. Let $\mathcal{D}_a$ be the set of all boxes at (i, j) with $i \leq a < j$. A *diagram* D on $\mathcal{D}_a$ is a finite collection of boxes in $\mathcal{D}_a$. For a diagram D on $\mathcal{D}_a$, we associate a graph with the vertex set $\mathbb{Z}$ obtained by adding an edge between i and j for each box at (i, j) in D. We say that a diagram D is a *connected tree* if the associated graph consists of all but finitely many isolated points and a single tree, and all vertices in the tree have distinct residues modulo n. Let $\mathrm{Supp}(D)$ be the set consisting of indices of all vertices in the single tree in the associated graph for D and $c(D)$ the number of vertices in the tree with index $\leq a$. Note that the box at $(i, i+np)$ does not appear in a connected tree for any $i, p \in \mathbb{Z}$.

A *labeling* D_L on a diagram D is a bijection from a set $\{1, 2, \ldots, |D|\}$ to the set of boxes in D. For a labeling L of a connected tree D, one can associate an element in affine FK algebra defined by $x_{D_L} = [D_L(1)][D_L(2)] \ldots [D_L(|D|)]$ where $[D_L(i)]$ is $[a_i b_i]$ for the i-th box placed at (a_i, b_i). We call two labelings L and L' are equivalent if we have $x_{D_L} = x_{D_{L'}}$ by only using commutation relation.

The following lemma is an obvious generalization of [30, Lemma7].

Lemma 4.6. *Let v, l be a positive integer and D be a connected tree in $\mathcal{D}_a$ with $l + v$ boxes contained in l rows and $v + 1$ columns. Then the following two sets are equal:*

(1) The classes of labelings of D such that the class contains a labeling with: $i_1, \ldots, i_l$ are distinct, $j_1 \leq \cdots, \leq j_l$, $j_{l+1}, \ldots, j_{l+v}$ are distinct, $i_{l+1} \leq \cdots \leq i_{l+v}$.

(2) The classes of labelings of D such that the class contains a labeling with: $i_1, \ldots, i_{l-1}$ are distinct, $j_1 \leq \cdots, \leq j_{l-1}$, $j_l, \ldots, j_{l+v}$ are distinct, $i_l \leq \cdots \leq i_{l+v}$.

Let $M(D) = \{D_{L_1}, \ldots, D_{L_h}\}$ be the set of representative labelings of equivalent classes in Lemma 4.6.

Definition 4.7. *Let m and a be positive integers. Define $\mathbf{p}_m(a)$ in $\widetilde{FK}_n$ by*

$$\mathbf{p}_m(a) = \sum_{D \in \mathcal{D}_a} \sum_{D_L \in M(D)} (-1)^{c(D)-1} x_{D_L}$$

where the first sum runs over all connected trees in $\mathcal{D}_i$.

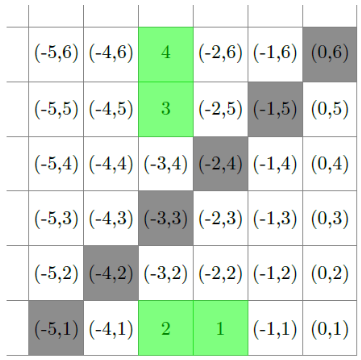

FIGURE 1. A connected tree on D_0 for $n = 6$. We color the box at (i, j) gray when $j - i = 0 \bmod n$, and color the box green when the box is in the diagram. In this example, the labeling of the green diagram satisfies Lemma 4.6 with $l = 3, v = 1$. We have $x_{D_L} = [-2, 1][-3, 1][-3, 5][-3, 6]$ with $c(D) = 2$.

Since all $[ij]$'s appearing in $\mathbf{p}_m(a)$ satisfy $i \le a < j$, $\mathbf{p}_m(a)$ gives a well-defined action on the affine nilCoxeter algebra by Corollary 3.3. We denote $\mathbf{p}_m(0)$ by $\mathbf{p}_m$.

Remark 4.8. *Roughly speaking, $\mathbf{p}_m(i)$ behaves similarly to the infinite summation $\sum_{j=-\infty}^{i} \tilde{\theta}_j^m$. The expression $\sum_{j=-\infty}^{i} \tilde{\theta}_j^m$ is not well-defined since we have relations $\tilde{\theta}_i = \tilde{\theta}_{i+n}$ and $\sum_{i=0}^{n-1} \tilde{\theta}_i = 0$ if we use all relations (a)-(e) defining affine Fomin-Kirillov algebra. Recall that $\mathcal{A}'$ be the quotient algebra of $\mathcal{A}$ modulo relations (a)-(c). Then $\mathbf{p}_m(a)$ as an element of $\mathcal{A}'$ is the same as*

$$\sum_{i=-\infty}^{a} \sum [ia_{i,1}] \dots [ia_{i,m}] = \lim_{j \to -\infty} \sum_{i=j}^{a} \sum [ia_{i,1}] \dots [ia_{i,m}]$$

where the second sum is over all integers $a_{i,1}, \dots a_{i,m}$ such that $a_{i,1}, \dots, a_{i,m}, i$ have distinct residues modulo n. Here we use $\sum [ia_{i,1}] \dots [ia_{i,m}]$ instead of $\tilde{\theta}_i^m$. However, the commutativity between $\mathbf{p}_m(a)$ and $\tilde{\theta}_i$ does not follow if we do not use the relations (d)-(e). In fact, $\tilde{\theta}_i$'s does not commute without the relation (d).

There is a surjection from the subalgebra generated by $\tilde{\theta}_i$ and $\mathbf{p}_m$ in $\mathcal{A}'$ to the subalgebra generated by Dunkl elements in the finite FK algebra. For a fixed set $T = (a_1 < \dots < a_p)$, one can define the projection $r_T : \mathcal{A}' \to \mathcal{A}'$ defined by $[ij] \mapsto [ij]$ if $\{i, j\} \subset T$ and 0 otherwise. For $T = \{1, 2, \dots, n\}$, this is the canonical surjection from $\mathcal{A}'$ to FK_n.
One can check that $r_T(\tilde{\theta}_i) = \theta_i$ for $1 \le i \le n$. The restriction of $\mathbf{p}_m(i)$ to finite Fomin-Kirillov algebra via r_T is the same as $p_m(\theta_1, \dots, \theta_i) = \theta_1^m + \dots + \theta_i^m$. Indeed, Mészáros, Panova and Postnikov gave a (positive) expression of the Schur function $s_\lambda(\theta_1, \dots, \theta_i)$ in $\theta_1, \cdots, \theta_i$ for the hook shape λ in [30], and one can deduce the expression for $\theta_1^m + \dots + \theta_i^m$ from the well-known identity

$$p_m = \sum_{i=0}^{m-1} (-1)^i s_{(m-i,1^i)}.$$

4.3. **More identities among Dunkl elements and MN elements.** For $p < n$, let $T = (a_1 < \cdots < a_p)$ be a set of distinct integers with distinct residues modulo n. Let f_T be an injection from $[n]$ to $\mathbb{Z}$ defined by $f_T(i) = a_i$. Then for such f_T, there is a injection from FK_n to $\mathcal{A}'$ defined by

$$[ij] \in FK_n \mapsto [f_T(i)f_T(j)] \in \mathcal{A}'.$$

Since all relations among $[ij] \in FK_n$ follows from relations (a)-(c) in $\widetilde{FK}_n$, this map is well-defined. Therefore, all relations between elements in FK_n also hold in $\mathcal{A}'$ and $\widetilde{FK}_n$ after applying the map f_T.

Theorem 4.9. *For $m < n, i \in \mathbb{Z}$, we have*

$$\mathbf{p}_m(i) + \tilde{\theta}^m_{i+1} = \mathbf{p}_m(i+1)$$

in $\widetilde{FK}_n$.

Proof. By Corollary 4.4, it is enough to show that $\mathbf{p}_m(i) + \sum [ia_1] \ldots [ia_m] = \mathbf{p}_m(i+1)$ where the sum is over all integers $a_1, \ldots, a_m, i$ having distinct residues modulo n. In fact, we will show this identity in $\mathcal{A}'$.
For a fixed support $T = (a_1 < \cdots < a_p)$, the restriction of the identity via r_T is

$$r_T(\mathbf{p}_m(i)) + \sum [ib_1] \ldots [ib_m] = r_T(\mathbf{p}_m(i+1)) \tag{5}$$

on the support T, where the sum runs over all $\{i, b_1, \cdots, b_m\} = T$ with $m+1 = p$. The identity (5) is the image of the identity $(\theta^m_1 + \cdots + \theta^m_i) + \theta^m_{i+1} = (\theta^m_1 + \cdots + \theta^m_i + \theta^m_{i+1})$ via the map f_T, therefore the identity holds for each support T. After taking the summation of the identity (5) for all possible supports T, the theorem follows. □

One can apply Theorem 4.9 to prove the following theorems.

Theorem 4.10. *For $m > 0$, we have*

$$\sum_{i=1}^{n} \tilde{\theta}^m_i = 0.$$

Proof. It is obvious from Theorem 4.9 and the fact $\mathbf{p}_m(0) = \mathbf{p}_m(n)$ by the relation (e). □

Theorem 4.11. *For $i, a \in I$,we have*

$$s_i \mathbf{p}_m(a) = \begin{cases} \mathbf{p}_m(a) & \textit{if } i \neq a \\ \mathbf{p}_m(i) + \tilde{\theta}^m_{i+1} - \tilde{\theta}^m_i. & i{=}a \end{cases} \tag{6}$$

Proof. If i and a are distinct modulo n, one can show that $s_i \mathbf{p}_m(a) = \mathbf{p}_m(a)$ by Definition 5.6. Indeed, this follows from the fact that for a connected tree D on $\mathcal{D}_a$ and $L \in M(D)$, $s_i(D)$ is also a connected tree on $\mathcal{D}_a$ and the labeling $s_i(L)$ satisfies Lemma 4.6 where $s_i(L)$ is the unique labeling on $s_i(D)$ satisfying $s_i(x_{D_L}) = x_{s_i(D)_{s_i(L)}}$ without the commutation relation. When $i = a$, by Theorem 4.9 we have

$$s_i \mathbf{p}_m(i) = s_i(\mathbf{p}_m(i-1) + \tilde{\theta}^m_i) = \mathbf{p}_m(i-1) + \tilde{\theta}^m_{i+1} = \mathbf{p}_m(i) + \tilde{\theta}^m_{i+1} - \tilde{\theta}^m_i.$$

□

5. Bruhat operators for MN elements

For $\mathbf{x} = \mathbf{p}_m$, we call $D_{\mathbf{p}_m}$ a *Murnaghan-Nakayama operator* of degree m (a MN operator in short). In this section, we investigate identities for MN operators (Theorem 5.3, 5.6, 5.7) that uniquely determine the MN operators. Computations in this section will be crucial proving main theorems.

Let $\mathbb{A}(\mathcal{E})$ be the smash product of $\mathbb{A}$ and $\mathcal{E}$ defined by the *equivariant Bruhat action*

$$A_w[ij] = \begin{cases} s_w([ij])A_w + A_{wt_{ij}} & \text{if } \ell(wt_{ij}) = \ell(w) - 1 \\ s_w([ij])A_w & \text{otherwise.} \end{cases} \tag{7}$$

To distinguish this action with the Bruhat action defined in (2), we will use $A_w \cdot [ij]$ for the Bruhat action. An element in $\mathbb{A}(\mathcal{E})$ can be written in a standard form i.e. in the form

$$\sum_{w \in \widetilde{S}_n} f_w A_w$$

where f_w is in $\mathcal{E}$ by the equivariant Bruhat action. One can show that $\mathbb{A}(\mathcal{E})$ has a basis $\{A_w \mid w \in \widetilde{S}_n\}$ over $\mathcal{E}$.

The equivariant Bruhat action strictly contains all information on the Bruhat action. Let $\phi : \mathbb{A}(\mathcal{E}) \to \mathbb{A}$ be the evaluation map at 0 defined by sending all $[ij]$ to 0. For example, $\phi(2[12][23]A_2 + 2[23]A_1 + 3A_0) = 3A_0$. Then we have $\phi(A_w[ij]) = A_w \cdot [ij]$ since $\phi([ij]A_w) = 0$ for all $[ij] \in \mathcal{S}$. Note that the Bruhat action is an action on the affine nilCoxeter algebra $\mathbb{A}$, but the equivariant Bruhat action is an action on $\mathbb{A}(\mathcal{E})$.

Let $\mathcal{C}$ be the subalgebra of $\mathcal{E}$ generated by all $\tilde{\theta}_i$'s and $\mathbf{p}_m$'s, and $\mathcal{C}_{Fl_n}$ be the subalgebra generated by Dunkl elements $\tilde{\theta}_i$. Let $\mathbb{A}_{\mathcal{C}}$ be the subalgebra of $\mathbb{A}(\mathcal{E})$ generated by all A_i's and $\mathcal{C}$, and let $\mathbb{A}_{\mathcal{C}_{Fl_n}}$ be the subalgebra of $\mathbb{A}(\mathcal{E})$ generated by all A_i's and $\mathcal{C}_{Fl_n}$.

5.1. MN operators of degree one.

Let us consider $D_{\mathbf{p}_1(a)}(A_w)$ for $w \in \widetilde{S}_n, a \in \mathbb{Z}$. From the definition of $\mathbf{p}_1(a)$ we have

$$D_{\mathbf{p}_1(a)}(A_w) = \sum_{w \to u} A_u \tag{8}$$

where the sum is over marked strong covers $w \to u$ with respect to a. It turns out this calculation also appears in the study of the *affine nilHecke algebra* [19, 29] and BSS operators.

Theorem 5.1. *For $w \in \widetilde{S}_n, 1 \le a \le n$, we have*

$$D_{\mathbf{p}_1(a)}(A_w) = D^{(a)}_{[1]}(A_w) = D_{s_a}(A_w)$$

where $D^{(a)}_{[1]}$ is the BSS operator for $[1]$ *and D_{s_0} is the cap operator for s_a.*

Proof. The first identity follows from Equation (8). The second identity follows from the identification of the BSS operators and the cap operators constructed in

[29] which we will discuss in Section 6. □

Note that Theorem 1.1 is the generalization of this identity for higher degree, and the first identity in Theorem 1.1 follows from Theorem 5.1.

5.2. **MN operators of higher degree.** In this section, we prove Theorem 5.3, 5.6, 5.7 which uniquely determine the MN operators.

Theorem 5.2. *For $a, j \in I$ and $m < n$, we have*

$$A_a \cdot \mathbf{p}_m(a) = (\mathbf{p}_m(a) - \tilde{\theta}_a^m)A_a + A_a \cdot \tilde{\theta}_a^m.$$

$$A_j \cdot \mathbf{p}_m(a) = (\mathbf{p}_m(a))A_j.$$

where $j \neq a$ modulo n.

Proof. The second identity follows from Theorem 4.11. By Theorem 4.9 and 4.11, we have

$$A_a \cdot \mathbf{p}_i(a) = A_a \cdot (\mathbf{p}_i(a-1) + \tilde{\theta}_a^m) = (\mathbf{p}_i(a) - \tilde{\theta}_a^m)A_a + A_a \cdot \tilde{\theta}_a^m.$$

□

The following theorem is a consequence from the second identity in Theorem 5.2.

Theorem 5.3. *For $w \in \widetilde{S}_n$ and $w' \in S_n$, we have*

$$D_{\mathbf{p}_m}(A_w A_{w'}) = D_{\mathbf{p}_m}(A_w)A_{w'}.$$

We need following lemmas to prove Theorem 5.6.

Lemma 5.4. *For $w \in \widetilde{S}_n$, $A_w \mathbf{p}_m - \mathbf{p}_m A_w$ lies in $\mathbb{A}_{\mathcal{C}_{Fl_n}}$.*

Proof. We use induction on $\ell(w)$. Let $w = vs_i$ with $\ell(w) = \ell(v) + 1$ and assume that the theorem holds for v. Then

$$\begin{aligned} A_w \mathbf{p}_m - \mathbf{p}_m A_w &= A_v A_i \mathbf{p}_m - \mathbf{p}_m A_v A_i \\ &= \begin{cases} A_v \mathbf{p}_m A_i - \mathbf{p}_m A_v A_i & \text{for } i \neq a \\ A_v \mathbf{p}_m A_a - \mathbf{p}_m A_v A_a + A_v(A_a \tilde{\theta}_a^m - \tilde{\theta}_a^m A_a) & \text{for } i = a. \end{cases} \end{aligned}$$

Note that we used Theorem 5.2 in the calculation. Since both $(A_v \mathbf{p}_m - \mathbf{p}_m A_v)A_i$ and $A_v(A_a \tilde{\theta}_a^m - \tilde{\theta}_a^m A_a)$ lie in $\mathbb{A}_{\mathcal{C}_{Fl_n}}$, the theorem follows. □

One may write $A_w \mathbf{p}_m - \mathbf{p}_m A_w = \sum_v f_{v,m}^w A_v$ where $f_{v,m}^w$ is a (noncommutative) polynomial in $\tilde{\theta}_i$ of degree $m - \ell(w) + \ell(v)$.

Lemma 5.5. *For $\mathbf{h} \in \mathbb{B}$, we have $\mathbf{h} \cdot \tilde{\theta}_i = 0$.*

Proof. By Theorem 4.9, it is enough to show that $\mathbf{h} \cdot \mathbf{p}_1(a)$ is independent of a. By Theorem 5.1, we have $\mathbf{h}_j \cdot \mathbf{p}_1(a) = D_{[1]}^{(a)}(\mathbf{h}_j)$. Since it is proved in [3] that the operators $D_{[1]}^{(a)}$ for different a's on $\mathbb{B}$ coincide, we are done. □.

Theorem 5.6. *For $\mathbf{h} \in \mathbb{B}$ and $w \in \widetilde{S}_n$,*

$$D_{\mathbf{p}_m}(\mathbf{h}A_w) = D_{\mathbf{p}_m}(\mathbf{h})A_w + \mathbf{h}D_{\mathbf{p}_m}(A_w).$$

Proof.

$$\begin{aligned}
D_{\mathbf{p}_m}(\mathbf{h}A_w) &= \phi(\mathbf{h}A_w \cdot \mathbf{p}_m) \\
&= \phi(\mathbf{h}\mathbf{p}_m A_w) + \phi(\mathbf{h}\sum_v f^w_{v,m} A_v) \\
&= \phi(\mathbf{h}\mathbf{p}_m) A_w + \phi(\mathbf{h} \sum_{\substack{v \\ \ell(v)=\ell(w)-m}} f^w_{v,m} A_v) \qquad \text{(By Lemma 5.5)} \\
&= D_{\mathbf{p}_m}(\mathbf{h}) A_w + \mathbf{h} D_{\mathbf{p}_m}(A_w).
\end{aligned}$$

□

Theorem 5.7. *For $1 \le m \le i < n$ and $a \in \mathbb{Z}$, we have*

$$D_{\mathbf{p}_m(a)} \mathbf{h}_i = \mathbf{h}_{i-m}.$$

Proof. It is enough to show the statement for $a = 0$ by taking an automorphism of $\mathbb{A}$ (resp. $\mathcal{E}$) by sending A_i to A_{i-a} (resp. $[ij]$ to $[i-a, j-a]$). Note that $\mathbf{h}_i$'s are invariant under the automorphism.

For $0 \in J \subset I$ with $|J| = i$, let J' be the connected subset $\{-j_1, -j_1 + 1, \dots, 0, 1, \dots, j_2\}$ of J with maximal size containing 0. The cyclically decreasing element w_J can be written of the form $w_{J'} w_{J''}$ where J is a disjoint union of J' and J''. By the construction of J' and J'', $w_{J'}$ and $w_{J''}$ commute with each other and $w_{J''}$ does not contain s_0. If J does not contain 0, we set $w_{J'} = id$ and $w_{J''} = w_J$. Note that $w_{J'}$ is simply $s_{j_2} s_{j_2 - 1} \dots s_{1-j_1} s_{-j_1}$ and the inversion set of $w_{J'}$ is $\{(-j_1, i) \mid -j_1 < i \le j_2 + 1\}$.

Let us calculate $A_{w_J} \cdot \mathbf{p}_m$. Since $w_{J''}$ does not contain s_0, we have

$$A_{w_J} \cdot \mathbf{p}_m = \left(A_{w_{J'}} A_{w_{J''}}\right) \cdot \mathbf{p}_m = \left(A_{w_{J'}} \cdot \mathbf{p}_m\right) A_{w_{J''}}.$$

Let $\mathbf{x} = [x_1 y_1] \cdots [x_m y_m]$ be the noncommutative monomial of degree m in $\mathcal{E}$ satisfying $x_b \le 0 < y_b$ for all b. Then one can show that $A_{w_{J'}} \cdot ([x_1 y_1] \cdots [x_m y_m])$ does not vanish if and only if $x_b = -j_1$ for all b and $j_2 + 1 \ge y_m > \cdots > y_1 \ge 0$. In this case, we have

$$A_{w_{J'}} \cdot ([x_1 y_1] \cdots [x_m y_m]) = A_{w_{J/\{y_1, y_2, \dots, y_m\}}}.$$

Consider the set of all pairs $\Omega = \{(A_{w_J}, \mathbf{x})\}$ where J is a subset of I with $|J| = i$ and $\mathbf{x} = [x_1 y_1] \cdots [x_m y_m]$ satisfying $x_b = -j_1$ for all b and $j_2 + 1 \ge y_m > \cdots > y_1 \ge 0$. Define the map from Ω to the set consisting of A_v's for all cyclically decreasing elements v of length $i - m$, by sending $(A_{w_J}, \mathbf{x})$ to $A_{w_{J/\{y_1, y_2, \dots, y_m\}}}$.

We claim that this map is bijective by providing an inverse map. For $K \subset I$ with $|K| = i - m$ and a nonnegative integer p, define K_p to be the union of K and $\{0, 1, \dots, p\}$. Let l be the minimal number satisfying $|K_l| = m$. Then we set $J = K_l$, $x_b = -j_1$ for all b, and $\{y_1 < \cdots < y_m\} = J/K$. One can show that both maps are inverse each other. Since all $\mathbf{x}$'s that occur in the set Ω appear with a coefficient 1 in $\mathbf{p}_m$, the theorem follows. □

6. Relations between operators

Recall that we abuse notation D_u so that it means one of Bruhat operators, cap operators, or BSS operators depending on u. For properties of cap operators and BSS operators, we will refer to [3, 4, 29].

Recall that for $0 \leq i < m < n$, let $\rho_{i,m}$ be $s_{-i}s_{-i+1}\dots s_{-1}s_{m-1-i}s_{m-2-i}\dots s_1s_0$ and $J_{i,m}$ be $[m-i, 1^i]$.

Theorem 6.1.

$$D_{\rho_{i,m}} = D_{J_{i,m}}.$$

Proof. Note that for $i = 0, m-1$, it follows from the theorems in [29]. Indeed, for $i = 0$ the theorem is the main theorem in [29], and for $i = m-1$ one can show the theorem by applying the automorphism of $\widetilde{S}_n$ and $\mathbb{A}$ sending s_i (resp. A_i) to s_{-i} (resp. A_{-i}).

For a composition $J = [j_1, j_2, \dots, j_l]$, observe from the definition of D_J that

$$D_J = D_{j_1} \circ D_{[j_2,\dots,j_l]} - D_{[j_1+j_2,\dots,j_l]}.$$

By setting $J = J_{i,m}$, we have

$$D_{J_{i,m}} = D_{[m-i]} \circ D_{[1^i]} - D_{J_{i-1,m}}.$$

Since it is already known that $D_{[m-i]} = D_{\rho_{0,m-i}}$ and $D_{[1^i]} = D_{s_{-i+1}s_{-i+2}\cdots s_{-1}s_0}$, it is enough to show that

$$D_{\rho_{i,m}} = D_{\rho_{0,m-i}} \circ D_{s_{-i+1}s_{-i+2}\cdots s_{-1}s_0} - D_{\rho_{i-1,m}}.$$

Since D_w can be identified with the Schubert class ξ^w, it is enough to show that

$$\xi^{\rho_{i,m}} = \xi^{\rho_{0,m-i}} \circ \xi^{s_{-i+1}s_{-i+2}\cdots s_{-1}s_0} - \xi^{\rho_{i-1,m}}.$$

The above equality can be shown in $\Lambda^{(k)} \cong H^*(\hat{Gr})$, since each Schubert class appearing above corresponds to a Schur function. In fact, the calculation of Schur functions $s_{J_{0,m-i}}s_{J_{i-1,i}} = s_{J_{i,m}} + s_{J_{i-1,m}}$ implies the above identity. □

Proof of Theorem 1.1. We show Theorem 1.1 by proving the identity

$$D_{\mathbf{p}_m} = \sum_{i=0}^{m-1} (-1)^i D_{J_{i,m}}.$$

First of all, they both satisfy the following lemma:

Lemma 6.2. *For $D = D_{\mathbf{p}_m}$ or $D_{J_{i,m}}$, $w \in \widetilde{S}_n$ and a 0-Grassmannian element v, we have*

$$D(A_wA_v) = D(A_w)A_v.$$

Proof. For $D = D_{\mathbf{p}_m}$ it follows from Theorem 5.3, and for $D = D_{J_{i,m}}$ it follows from the fact that $D_{J_{i,m}}$ are generated by $D_{[a]}$ for $a \geq 1$ and the lemma holds for $J = [a] = J_{0,a}$ by [4, Theorem 4.8]. □

For any composition J, the restriction of D_J to $\mathbb{B}$ is $\overline{s_J}^{\perp}$ where s_J is the ribbon Schur function indexed by J [4, Theorem 4.9]. By letting $J = J_{i,m}$, the restriction

of $D_{J_{i.m}}$ is $\overline{s_{J_{i,m}}}^{\perp}$, where $s_{J_{i,m}}$ is the Schur function for the hook shape $[m-i, 1^i]$. Recall the following theorems about the power sum symmetric functions p_m: (see [33] for instance):

$$p_m = \sum_{i=0}^{m-1} (-1)^i s_{J_{i,m}},$$

$$p_m^{\perp}(fg) = p_m^{\perp}(f)g + fp_m^{\perp}(g),$$

$$p_m^{\perp}(h_i) = h_{i-m}$$

for any symmetric functions f, g. Therefore, we proved that $D = \sum_{i=0}^{m-1}(-1)^i D_{J_{i,m}}$ satisfies the following identities.

(1) $D(fg) = D(f)g + fD(g)$ for $f, g \in \mathbb{B}$.
(2) $D(hA_v) = D(h)A_v$ for $h \in \mathbb{B}$ and a 0-Grassmannian element v.
(3) $D(\mathbf{h}_i) = \mathbf{h}_{i-m}$

Note that the above identities uniquely determine D as an action on $\mathbb{A}$. Since $\mathbf{p}_m$ also satisfies the above identities by Theorem 5.3, 5.6, 5.7, the main theorem follows. $\square$

7. Divided difference operators

In this section, we define the divided difference operators on the affine FK algebra and describe the connection between the divided difference operators and BGG operators.

7.1. BGG operators and the affine nilCoxeter algebra.

For $i \in \mathbb{Z}/n\mathbb{Z}$, let $p_i : \hat{Fl} \to \hat{Fl}^{(i)}$ be the projection of $\hat{Fl}$ onto the partial flag scheme defined by the minimal parabolic subgroup corresponding to i. Then the BGG operator $\partial_i : H^*(\hat{Fl}) \to H^*(\hat{Fl})$ is defined by $\partial_i := p_i^* p_{i*}$. This definition is equivalent to the following.

Definition 7.1. *For $i \in \widetilde{S}_n$ Let ξ^w be the Schubert class for w in $H^*(\hat{Fl})$. Then*

$$\partial_i \xi^w = \xi^{ws_i} \text{ if } \ell(ws_i) = \ell(w) - 1, = 0 \text{ otherwise.}$$

For $w = s_{i_1} \dots s_{i_l}$, ∂_w is defined by $\partial_{i_1} \dots \partial_{i_l}$. One can show that ∂_i's and elements A_i in the affine nilCoxeter algebra satisfy the same relations, so that we will identify ∂_w and A_w. Under this identification, there is an action of the affine nilCoxeter algebra on $H^*(\hat{Fl})$ by $A_w \bullet f := \partial_w f$, for $f \in H^*(\hat{Fl})$. We call this action $\bullet$-action. When considering f as an element in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$, we have

$$(A_i \bullet f)(A_w) = f(A_w A_i).$$

See [19], [27, Chapter 4] for details.

7.2. **Divided difference operators on $\widetilde{FK}_n$.** Recall that for $w \in \widetilde{S}_n$, the action of w on $\widetilde{FK}_n$ is given by $w[ij] = [w(i)w(j)]$.

For integers $i < j$ with distinct residues mod n, let Δ_{ij} be the divided difference operator as a left action on $\widetilde{FK}_n$ defined by the following:

(1) For $\mathbf{x}, \mathbf{y} \in \widetilde{FK}_n$, $\Delta_{ij}(\mathbf{xy}) = \Delta_{ij}(\mathbf{x})\mathbf{y} + t_{ij}(\mathbf{x})\Delta_{ij}(\mathbf{y})$.

(2) $\Delta_{ij}([ab]) = 1$ when $[ij] = [ab]$ in $\widetilde{FK}_n$, and 0 otherwise.

Note that the definition of Δ_{ij} generalizes the operators denoted by the same in [8]. The divided difference operator Δ_{ij} is well-defined because Δ_{ij} stabilizes the two-sided ideal generated by relations (a)-(e), which is easy to check.

Recall the map $D : \mathcal{E} \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ defined by the Bruhat operator. By considering the restriction $\mathcal{C} \subset \mathcal{E}$, we have $D : \mathcal{C} \to H^*(\hat{Fl}) \subset \mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$. By applying $\phi_{id,*}$, one can define a map $D^{id} := \phi_{id,*} \circ D : \mathcal{C} \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q}) \cong H^*(\hat{Fl})$ defined by $D^{id}_w = \phi_{id,*} \circ D_w$. For the rest of the section, we show that the divided difference operator $\Delta_i := \Delta_{i,i+1}$ on $\mathcal{C}$ gives an induced divided difference operator Δ_i on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$, and the induced operator is the same as the action $A_i \bullet$.

Lemma 7.2. *Let $w \in \widetilde{S}_n$ and $\mathbf{x} \in \mathcal{E}$. In $\mathbb{A}(\mathcal{E})$, we have*

$$A_i \mathbf{x} = s_i(\mathbf{x}) A_i + \Delta_i(\mathbf{x}).$$

Proof. The lemma follows from the definition of the equivariant Bruhat action. □

Corollary 7.3.

$$D^{id}_{\Delta_i(\mathbf{x})}(A_w) = D^{id}_{\mathbf{x}}(A_w A_i) = \left(A_i \bullet D^{id}_{\mathbf{x}}\right)(A_w).$$

Proof. By Lemma 7.2, we have

$$D_{\Delta_i(\mathbf{x})}(A_w) = D_{\mathbf{x}}(A_w A_i) - D_{s_i(\mathbf{x})}(A_w) A_i.$$

By applying $\phi_{id,*}$, the corollary follows because $\phi_{id,*}(D_{s_i(\mathbf{x})}(A_w) A_i) = 0$. □

Note that Corollary 7.3 implies Theorem 1.4 stating that the divided difference operators are the same as BGG operators.

8. Affine Schubert polynomial

In this section, we define the affine Schubert polynomials using the divided difference operators.

Recall the isomorphisms

$$\begin{aligned} H^*(\hat{Fl}) &\cong \mathcal{R}_{\hat{Gr}} \otimes_{\mathbb{Q}} \mathcal{R}_{Fl_n} \\ &\cong R_n := \Lambda^{(k)} \otimes \mathbb{Q}[x_1, \cdots, x_n]/\{s_i(x) = 0 \quad \forall i\}. \end{aligned}$$

Recall that $\Lambda^{(k)}$ is generated by $\overline{p_m}$, which we denote by p_m.

The divided difference operators ∂_i on $H^*(\hat{Fl})$ and the Schubert class ξ^w correspond to the divided difference operators on R_n defined in Definition 1.3 and the

Schubert polynomials respectively. Therefore, the following definition of the Schubert polynomials is justified as polynomial representatives of the Schubert class ξ^w.

Definition 8.1 (Theorem 1.5)**.** *For $w \in \widetilde{S}_n$, the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ is the unique homogeneous element of degree $\ell(w)$ in R_n satisfying*

$$\partial_i \tilde{\mathfrak{S}}_w = \begin{cases} \tilde{\mathfrak{S}}_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $i \in \mathbb{Z}/n\mathbb{Z}$, with the initial condition $\tilde{\mathfrak{S}}_{id} = 1$.

Note that the divided difference operators ∂_i for nonzero i restricted on $R_n^{Fl_n} = \mathbb{Q}[x_1, \cdots, x_n]/\{s_i(x) = 0 \quad \forall i\}$ is the same as the divided difference operators defined by Lascoux and Schützenberger [24]. This implies that the affine Schubert polynomial for $w \in S_n$ is just the Schubert polynomial for w. Moreover, the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ for 0-Grassmannian element w is the same as the affine Schur functions, and for $w \in \widetilde{S}_n$ the projection from R_n to R_n^{Gr} sends $\tilde{\mathfrak{S}}_w$ to the affine Stanley symmetric functions $\tilde{F}_w$ [22].

The affine Schubert polynomials can be computed from the affine Schur functions. For $w \in \widetilde{S}_n$, let v be an element in $\widetilde{S}_n$ such that wv is 0-Grassmannian with $\ell(wv) = \ell(w) + \ell(v)$. There is always such a v for any w [31]. Let $\tilde{F}_{wv}$ be the affine Schur function for wv. Then we have

$$\tilde{\mathfrak{S}}_w = \partial_{v^{-1}} \tilde{\mathfrak{S}}_{wv} = \partial_{v^{-1}} \tilde{F}_{wv}.$$

Note that there is a formula for the expansion of the affine Schur functions in terms of power sum symmetric functions [2], so that one can compute the affine Schubert polynomials from Definition 1.3, 1.5.

8.1. **Examples.** $n = 2$ case: For a positive integer a and $i \in \{0, 1\}$, define $w_{a,i}$ be the unique i-Grassmannian element of length a. Then

$$\begin{aligned} \tilde{\mathfrak{S}}_{w_{a,0}} &= m_{1,1,\ldots,1} = \frac{p_1^n}{n!} \\ \tilde{\mathfrak{S}}_{w_{a,1}} &= \frac{p_1^n}{n!} + \frac{p_1^{n-1}}{(n-1)!} x_1 \end{aligned}$$

$n = 3$ case:

$$\begin{aligned} \tilde{\mathfrak{S}}_{id} &= 1 \\ \tilde{\mathfrak{S}}_{s_0} &= p_1 \\ \tilde{\mathfrak{S}}_{s_1} &= p_1 + x_1 \\ \tilde{\mathfrak{S}}_{s_2} &= p_1 + x_1 + x_2 \\ \tilde{\mathfrak{S}}_{s_1 s_0} &= \frac{1}{2}(p_1^2 + p_2) \\ \tilde{\mathfrak{S}}_{s_2 s_1} &= \frac{1}{2}((p_1 + x_1)^2 + (p_2 + x_1^2)) \\ \tilde{\mathfrak{S}}_{s_2 s_1 s_0} &= \frac{1}{3} p_3 + \frac{1}{2} p_2 p_1 + \frac{1}{6} p_1^3. \end{aligned}$$

9. Nonnegativity conjecture

In this section, we discuss a generalization of Theorem 1.2 and the nonnegativity conjecture for Schubert polynomials in the Fomin-Kirillov algebra [8].

Although arguments in Section 6 show that the map D from $\mathcal{C}$ to $\mathcal{R}$ is surjective, there is strong evidence that D is in fact an isomorphism. First of all, all relations between Bruhat operators $D_{\tilde{\theta}_i}$ for Dunkl elements are proved at the level of the affine FK algebra, namely the commutativity between Dunkl elements (Theorem 4.2) and the vanishing of symmetric functions evaluated at Dunkl elements (Theorem 4.10). To show that D is an isomorphism, we need to show the following conjecture.

Conjecture 9.1. *The commutativity between Dunkl elements and MN elements follows from the relations (a)-(e). This implies that the map $D : \mathcal{C} \to \mathcal{R} \cong H^*(\hat{Fl})$ is an isomorphism.*

The strong evidence is that the MN element $\mathbf{p}_m(i)$ should be considered as $\sum_{j=-\infty}^{i} \tilde{\theta}_j^m$ secretly (see Remark 4.8). The infinite summation $\sum_{j=-\infty}^{i} \tilde{\theta}_j^m$ does not make sense in $\widetilde{FK}_n$ but make sense in $\mathcal{A}'$, since the relations (a)-(c) are compatible with the topology of A (Definition 3.1) but relations (d)-(e) are not compatible. If there is a way to make the expression $\sum_{j=-\infty}^{i} \tilde{\theta}_j^m$ valid while keeping the commutativity between Dunkl elements $\tilde{\theta}_i$, Conjecture 9.1 automatically follows.

Without the above argument, it is very hard to avoid a case-by-case proof for proving the commutativity of those elements from the relations (a)-(e). Note that Berg et. al. showed that BSS operators commute in [4] but their result relies heavily on the affine insertion [25] which involves complicated case-by-case algorithm and proofs.

Let $\tilde{\mathfrak{S}}_w^{\widetilde{FK}_n}$ be the *affine Schubert element* in $\mathcal{C}$ corresponding to the Schubert class ξ^w, assuming Conjecture 9.1. Then one can generalize the nonnegativity conjecture in [8].

Conjecture 9.2 (Nonnegativity conjecture for the affine Schubert elements)**.** *For $w \in \widetilde{S}_n$, the affine Schubert element $\tilde{\mathfrak{S}}_w^{\widetilde{FK}_n}$ can be written as a nonnegative linear combination of noncommutative monomials in $\widetilde{FK}_n$. Moreover, there exists a linear combination of noncommutative monomials with nonnegative integer coefficients.*

Conjecture 9.2 not only implies the nonnegativity conjecture in [8], but also implies the nonnegativity conjecture for the affine Schur elements and for the affine Stanley elements in $\mathcal{C}$ corresponding to affine Schur functions and affine Stanley symmetric functions respectively. Indeed, for 0-Grassmannian element w the affine Schubert element is the affine Schur element, and it is known that the affine Stanley symmetric functions are nonnegative linear combination of affine Schur functions [23].

A combinatorially nonnegative formula for the affine Schubert element would provide a combinatorial rule for the structure constants of the affine Schubert polynomials, as well as Littlewood-Richardson coefficients for the flag variety. Assume

that Conjecture 9.2 is true for $u \in \widetilde{S}_n$. Then $\tilde{\mathfrak{S}}_u^{\widetilde{FK}_n}$ can be written as $\sum a_{\mathbf{x}}\mathbf{x}$ where $\mathbf{x}$'s are noncommutative monomials and $a_{\mathbf{x}} \geq 0$. Since Bruhat operators and cap operators can be identified, we have

$$\sum_{\mathbf{x}} a_{\mathbf{x}}(A_w \cdot \mathbf{x}) = D_{\tilde{\mathfrak{S}}_w^{\widetilde{FK}_n}}(A_v) = D_u(A_w) = \sum_{v \in \widetilde{S}_n} p_{v,u}^w A_v.$$

for $w \in \widetilde{S}_n$. Then the structure coefficient $p_{v,u}^w$ is the same as the sum of $a_{\mathbf{x}}$ where $A_w \cdot \mathbf{x} = A_v$ which is nonnegative by Conjecture 9.2.

10. Murnaghan-Nakayama rule for the affine flag variety

In this section, we define a k-strong-ribbon and use them to combinatorially state the Murnaghan-Nakayama rule (MN rule in short) for the affine flag variety and the affine Stanley symmetric functions. Strong strips appearing in this section are strong strips with respect to 0.

For $w, u \in \widetilde{S}_n$ with $w > u$, consider a chain of marked strong covers from w to u

$$S = \left(w = w_0 \overset{(x_1,y_1)}{\longrightarrow} w_1 \overset{(x_2,y_2)}{\longrightarrow} \cdots \overset{(x_m,y_m)}{\longrightarrow} w_m = u\right).$$

Let l be the index of the chain defined by $((x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m))$ and $\mathbf{x}_l$ be the corresponding element $[x_1, y_1][x_2, y_2] \cdots [x_m, y_m]$ in the affine FK algebra. We denote the chain by $w \overset{l}{\rightsquigarrow} u$.

For $w, u \in \widetilde{S}_n$ with $w > u$, the chain $S = (w \overset{l}{\rightsquigarrow} u)$ is called a *k-strong-ribbon* when $\mathbf{x}_l$ is a term in the definition of $\mathbf{p}_m$, i.e., there exists a connected tree D on $\mathcal{D}_0$ and a labeling $L \in M(D)$ such that $\mathbf{x}_l = \mathbf{x}_{D_L}$. In this setup, we define the *sign* $\sigma(D)$ of the k-strong-ribbon by $(-1)^{c(D)-1}$. Since $\mathbf{x}_{D_L}$ uniquely determines the diagram D, we use $\sigma(l)$ instead of $\sigma(D)$. We let inside$(S) = w$, outside$(S) = u$ For $w, v \in \widetilde{S}_n$ and $m < n$, let $c_{m,v}^w$ be the number $\sum \sigma(l)$ where the sum runs over all k-strong-ribbons $(w \overset{l}{\rightsquigarrow} u)$. One can show from the definition that $D_{\mathbf{p}_m}(A_w) = \sum_u c_{m,u}^w A_u$.

Recall that ξ^w is the Schubert class for w in the cohomology of the affine flag variety and $\xi(m) = \sum_{i=0}^m (-1)^i \xi^{\rho_{i,m}}$. Note that $\xi(m)$ maps to p_m via the map $p_1^* : H^*(\hat{Fl}) \to H^*(Gr) \cong \Lambda^{(k)}$.

Theorem 10.1. *For $v \in \widetilde{S}_n$, we have*

$$\xi(m)\xi^v = \sum_{w \in \widetilde{S}_n} c_{m,v}^w \xi^w = \sum \sigma(l)\xi^w$$

where the second sum runs over all k-strong-ribbons $(w \overset{l}{\rightsquigarrow} u)$ from w to u.

Proof. For $u \in \widetilde{S}_n$, let ξ_u be the Schubert class for u in the homology of the affine flag variety and let $\langle \cdot, \cdot \rangle$ be the pairing between the cohomology and homology of the affine flag variety. Then we have

$$\langle \xi(m) \cup \xi^v, \xi_w \rangle = \langle \xi^v, D_{\mathbf{p}_m}(\xi_w) \rangle = c_{m,v}^w.$$

□

One can also obtain the MN rule for the affine Stanley symmetric functions. Lam showed that the Stanley symmetric function $\tilde{F}_w$ is the pullback $p_1^*(\xi^w)$. By applying the pullback p_1^* to both sides of Theorem 10.1, we have the following MN rule.

Corollary 10.2.

$$p_m\tilde{F}_v = \sum_{w\in\widetilde{S}_n} c^w_{m,v}\tilde{F}_w = \sum \sigma(l)\tilde{F}_w$$

where the second sum runs over all k-strong-ribbons ($w \overset{l}{\rightsquigarrow} v$) from w to v.

Example 10.3. *Consider the identity $\tilde{F}_{10}p_3 = \tilde{F}_{12310} - \tilde{F}_{20310} + \tilde{F}_{03210}$. Each term can be computed from the Bruhat actions of the following terms in $\mathbf{p}_m(0)$.*

$$\begin{aligned} s_1s_2s_3s_1s_0 \cdot [-2,1][-4,1][-1,1] &= s_1s_0 \\ s_2s_0s_3s_1s_0 \cdot [-4,1][-1,2][-1,1] &= s_1s_0 \\ s_0s_3s_2s_1s_0 \cdot [0,6][0,5][0,3] &= s_1s_0. \end{aligned}$$

and the corresponding $c(l)$, the number of non-positive integers appearing in the support, are $3,2,1$ respectively and the corresponding signs $\sigma(l)$ are $1,-1,1$ respectively.

11. Concluding remark

The connection between MN operators and BSS operators enables us to make a marking-free definition of strong strips and k-Schur functions. Indeed, since the MN element $\mathbf{p}_m(0)$ as an action on $\mathbb{B}$ is the same as $\overline{p_m}$ and $D_{\rho_{0,m}}$ is the same as $\overline{h_m}$, one can deduce the relation between $D_{\mathbf{p}_m}$ and $D_{\rho_{0,m}}$ from the relation between p_m and h_m in the ring of symmetric functions. Note that the formula for $D_{\rho_{0,m}}$ as an element in the affine FK algebra is much more complicated than the definition of $\mathbf{p}_m$. Indeed, this situation also happens in the finite Fomin-Kirillov algebra. The fact that $\mathbf{p}_m$ has fewer terms than other elements of the same degree makes computation substantially simpler.

One can give a new definition of k-Schur functions from the MN rule for the affine Stanley symmetric functions. A *k-strong-ribbon tableau* is a sequence $T = (S_1, S_2, \ldots)$ of k-strong-ribbons S_i such that $\mathrm{outside}(S_j) = \mathrm{inside}(S_{j+1})$ for all $j \in \mathbb{Z}_{>0}$ and $\mathrm{size}(S_i) = 0$ for all sufficiently large i. We define $\mathrm{inside}(T) = \mathrm{inside}(S_1)$ and $\mathrm{outside}(T) = \mathrm{outside}(S_i)$ for i large. The *weight* $wt(T)$ of T is the sequence

$$wt(T) = (\mathrm{size}(S_1), \mathrm{size}(S_2), \ldots).$$

Let $\sigma(T)$ be the product of $\sigma(l_i)$ for all i where l_i is the index of S_i.

By applying Theorem 10.2 repeatedly from $v = id$, one can deduce

$$p_\lambda = \sum_{v,T} \sigma(T)\tilde{F}_v$$

where the sum runs over all k-strong-ribbon tableaux T starting from v to id with weight λ. By dualizing the equality, we can get the formula for k-Schur functions

in terms of p_m.
Consider $\langle s_u^{(k)}, p_\lambda\rangle$ where $\langle \cdot,\cdot\rangle$ is the Hall inner product on $\Lambda_{(k)} \times \Lambda^{(k)}$. We have

$$\langle s_u^{(k)}, p_\lambda\rangle = \langle s_u^{(k)}, \sum_T \sigma(T)\tilde{F}_v\rangle = \sum \sigma(T).$$

where the last sum runs over all k-strong-ribbon tableaux T from u to id of weight λ. By using the identity $\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda$ where $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$ for $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$, we have

Theorem 11.1.

$$s_u^{(k)} = \sum_{\lambda,T} \frac{\sigma(T)}{z_\lambda} p_\lambda$$

where the sum runs over all k-strong-ribbon tableaux T from u to id of weight λ.

It would be interesting to generalize Theorem 11.1 to define t-dependent k-Schur functions by introducing t-dependent k-strong-ribbon tableaux.

Since the k-Schur function $s_u^{(k)}$ is known to be Schur-positive [26], there exists a S_n-representation whose Frobenius image is $s_u^{(k)}$. By Theorem 11.1, the characters of this representation is $\sum \sigma(T)$ where T runs over all k-strong-ribbon tableaux T from u to id of weight λ. Note that Chen and Haiman [6] conjectured t-graded S_n-representation whose Frobenius image is the involution of the t-dependent k-Schur function $\omega s_u^{(k)}[X;t]$. It would be interesting if the combinatorics studied by Chen and Haiman is related to the k-strong-ribbon tableaux in this paper.

Acknowledgments

I thank Thomas Lam, Leonardo Mihalcea, Mark Shimozono for helpful discussions.

References

[1] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys 28, no. 3 (1973) 126.
[2] J. Bandlow, A. Schilling, M. Zabrocki, *The Murnaghan-Nakayama rule for k-Schur functions*, J. Comb. Theory. Ser. A, 118, no. 5(2011), 1588-1607.
[3] C. Berg, F. Saliola, L. Serrano, *The down operator and expansions of near rectangular k-Schur functions*, J. Comb. Theory. Ser. A 120(2013), no. 3, 623-636.
[4] C. Berg, F. Saliola, L. Serrano, *Pieri operators on the affine nilCoxeter algebra*, Trans. Amer. Math. Soc. 366(2014), 531-546.
[5] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics, Vol. 231.
[6] L. Chen, M. Haiman, *A representation-theoretic model for k-atoms*, Talk 1039-05-169 at the AMS meeting in Claremont, McKenna, May 2008.
[7] A. Dalal, J. Morse, *Quantum and affine Schubert calculus and Macdonald polynomials*, arXiv:1402.1464.
[8] S. Fomin, A.N.Kirillov, *Quadratic algebras, Dunkl elements, and Schubert calculus*, Adv. Geom., Progress in Mathematics, 172(1999), 147-182.
[9] S. Fomin, R. Stanley, *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. 103(1994), no. 2, 196207.
[10] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. 109(2001), no. 3, 599-614.
[11] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+ 204 pp.

[12] M. Kashiwara, M. Shimozono, *Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials*, Duke Math. J. 148 (2009), no. 3, 501-538.
[13] A. Kirillov, *On some quadratic algebras $I\frac{1}{2}$: Combinatorics of Dunkl and Gaudin elements, Schubert, Grothendieck, Fuss-Catalan, universal Tutte and Reduced polynomials*, preprint RIMS-1817, 2015.
[14] A. Kirillov, T. Maeno, *Noncommutative algebras related with Schubert calculus on Coxeter groups*, European Journal of Combinatorics, 25 (2004), 1301-1325. Preprint RIMS-1437, 2003, 23p.
[15] A. Kirillov, T. Maeno, *A note on quantization on Nichols algebra model for Schubert calculus on Weyl groups* Lett. in Math. Phys. 72 (2005) 233-241. Preprint RIMS-1481, 2004, 8p.
[16] A. Kirillov, T. Maeno, *On some noncommutative algebras related with K-theory of flag varieties*, IRMN 60 (2005), 3753-3789, Preprint RIMS, 2005, 25p.
[17] A. Kirillov, T. Maeno, *Nichols-Woronowicz model of coinvariant algebra of complex reflection groups*, J. Pure Appl. Algebra 214 (2010), 402-409
[18] A. Kirillov, T. Maeno, *Affine nil-Hecke algebras and braided differential structure on affine Weyl groups*, Publ. RIMS, no. 48 (2012), 215-228.
[19] B. Kostant, S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Adv. Math. 62 (1986), no. 3, 187-237.
[20] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, 204. Birkh́auser Boston, Inc., Boston, MA, 2002. xvi+606 pp.
[21] T. Lam, *Affine Stanley symmetric functions*, Amer. J. Math. 128(6):1553-1586, 2006.
[22] T. Lam, *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. 21(2008), no.1, 259-281.
[23] T. Lam, *Affine Schubert classes, Schur positivity, and combinatorial Hopf algebras*, Bulletin of the LMS, 2011; doi:10.1112/blms/bdq110.
[24] A. Lascoux, M. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris. Sér. I Math. 294 (13): 447-450 (1982).
[25] T. Lam, L. Lapointe, J. Morse, M. Shimozono, *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. 208 (2010), no. 977.
[26] T. Lam, L. Lapointe, J. Morse, M. Shimozono, *k-shape poset and branching of k-Schur functions*, Mem. Amer. Math. Soc., 223 (2013), no. 1050.
[27] T. Lam, L. Lapointe, J. Morse, A. Schilling, M. Shimozono, M. Zabrocki, *k-Schur functions and affine Schubert calculus*, Fields Institute Monographs 33 (2014), VIII. 219, p.126 illus.
[28] T. Lam, S. Lee, M. Shimozono, in preperation.
[29] S. Lee, *Pieri rule for the affine flag variety*, arXiv:1406.4246v2.
[30] K. Mészáros, G. Panova, A. Postnikov, *Schur times Schubert via the Fomin-Kirillov algebra*, Electr. J. Comb. Vol 21, Issue 1 (2014)
[31] J. Morse, A. Schilling, *Crystal Approach to Affine Schubert Calculus*, Int. Math. Res. Notices., doi:10.1093/imrn/rnv194.
[32] D. Peterson, Lecture Notes at MIT, 1997.
[33] R. Stanley, *Enumerative Combinatorics*, Cambridge University Press, Vol. 2(1999), ISBN 0-521-56069-1.

School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul 130-722, Republic of Korea.
E-mail address: `lsjin@kias.re.kr`